Solutions

Mock Midterm 2

EECS 245, Fall 2025 at the University of Michigan

Name: ______________________________

uniqname: ______________________________

UMID: ______________________________

Room: ◯ 1365 LCSIB ◯ 2901 BBB

Instructions

- This exam consists of 8 problems, worth a total of (N/A) points, spread across 14 pages (7 sheets of paper).
- You have 80 minutes to complete this exam, unless you have extended-time accommodations through SSD.
- Write your uniqname in the top right corner of each page.
- For free response problems, you must show all of your work (unless otherwise specified), and circle your final answer. We will not grade work that appears elsewhere, and you may lose points if your work is not shown.
- For multiple choice problems, completely fill in bubbles and square boxes; if we cannot tell which option(s) you selected, you may lose points.
  ◯ A bubble means that you should only select one choice.
  ☐ A square box means you should select all that apply.
- You may refer to a single two-sided handwritten notes sheet. Other than that, you may not refer to any other resources or technology during the exam (no phones, watches, or calculators).

You are to abide by the University of Michigan/Engineering Honor Code. To receive a grade, please sign below to signify that you have kept the Honor Code pledge.

I have neither given nor received aid on this exam, nor have I concealed any violations of the Honor Code.

This page has been intentionally left blank.

Problem 1: Inversion

Let $A = \begin{bmatrix} 2 & -3 \\ 8 & -10 \end{bmatrix}$.

a) Find $\det(A)$, the determinant of A.

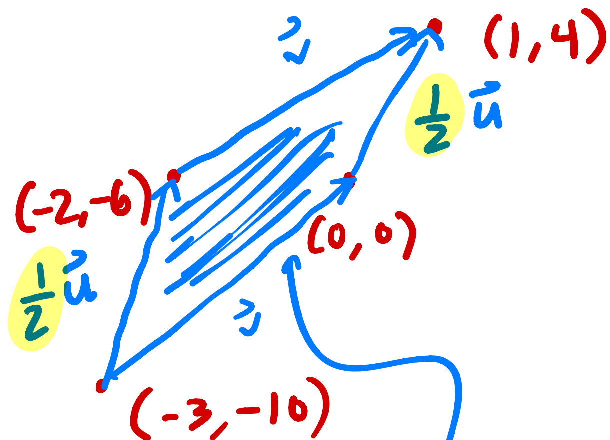
$\det(A) = \boxed{4}$

$2(-10) - 8(-3)$
$= -20 + 24$
$= 4$

b) Find the area of the four-sided polygon in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,4)$, $(-2,-6)$, and $(-3,-10)$.

Area $= \boxed{2}$

If A's cols are $\vec{u}$, $\vec{v}$ then



half the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$, so half the determinant

$\rightarrow \frac{4}{2} = 2$

c) Find A^{-1}, the inverse of A.

$A^{-1} = \boxed{\begin{bmatrix} -5/2 & 3/4 \\ -2 & 1/2 \end{bmatrix}}$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} -10 & 3 \\ -8 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -5/2 & 3/4 \\ -2 & 1/2 \end{bmatrix}$$

Big idea: → row space and null space are orthogonal

→ col space and left null sp orthogonal

Problem 2: True or False?

Let A be an $n \times d$ matrix. For each statement in this problem, select True or False. For True statements, provide a short justification; for False statements, provide a counterexample (or a general argument for why the statement is false).

a) If A is square, and $\vec{x} \in \text{nullsp}(A)$ and $\vec{y} \in \text{colsp}(A)$, then $\vec{x} \cdot \vec{y} = 0$.

○ True ● False

False in general.

e.g. $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ in nullsp(A)

$\vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in colsp(A)

$\vec{x} \cdot \vec{y} = -1 \neq 0$

b) If A is square, and $\vec{x} \in \text{nullsp}(A)$ and $\vec{y} \in \text{colsp}(A^T)$, then $\vec{x} \cdot \vec{y} = 0$.

● True ○ False

Let $A\vec{x} = \vec{0}$ and $\vec{y} = A^T\vec{b}$ by definition.

Then, $\vec{x} \cdot \vec{y} = \vec{x}^T(A^T\vec{b}) = \vec{x}^T A^T \vec{b}$

$= (A\vec{x})^T\vec{b} = \vec{0}^T\vec{b} = 0$

c) If A is symmetric ($A = A^T$), and $\vec{x} \in \text{nullsp}(A)$ and $\vec{y} \in \text{colsp}(A)$, then $\vec{x} \cdot \vec{y} = 0$.

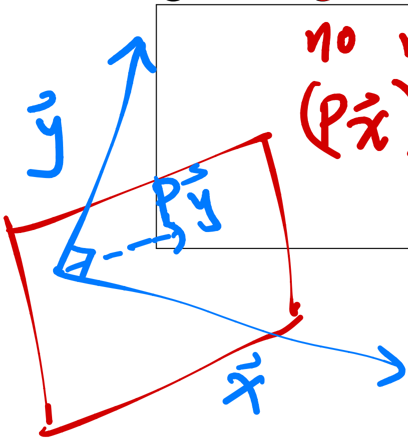
● True ○ False

If A symmetric, colsp(A) = colsp(A^T), so this is the same as the case above!

d) If P projects vectors in $\mathbb{R}^n$ onto the column space of A, and $\vec{x}, \vec{y} \in \mathbb{R}^n$ are orthogonal, then $P\vec{x}$ and $P\vec{y}$ are also orthogonal.

$P = A(A^TA)^{-1}A^T$ symmetric, idempotent

○ True ● False



no reason to believe this. but also,

$(P\vec{x}) \cdot P\vec{y} = (P\vec{x})^T(P\vec{y}) = \vec{x}^T P^T P \vec{y} = \vec{x}^T P \vec{y}$

no guarantee this is 0, even if $\vec{x}^T\vec{y} = 0$

note $P^T = P$ and $P^2 = P$

so $P^TP = P$

Problem 3: CR decomposition

Suppose $\vec{x}, \vec{y} \in \mathbb{R}^5$ and let

$$A = \begin{bmatrix} 3 & 0 & | & | & 3 \\ 1 & 1 & | & | & 2 \\ -1 & 1 & \vec{x} & \vec{y} & 0 \\ 0 & 1 & | & | & 1 \\ 2 & 0 & | & | & 2 \end{bmatrix}$$

col 5 = 1(col 1) + 1(col 2)

Additionally, suppose A has the following CR decomposition:

$$A = \underbrace{\begin{bmatrix} 3 & 0 \\ 1 & 1 \\ -1 & 1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}}_{5 \times r}^{C} \underbrace{\begin{bmatrix} 1 & 0 & 3 & -1 & b_1 \\ 0 & 1 & 0 & 2 & b_2 \end{bmatrix}}_{r \times 5}^{R}$$

$\rightarrow r = \text{rank} = 2$

a) Determine the following values.

rank$(A) = \boxed{2}$, $b_1 = \boxed{1}$, $b_2 = \boxed{1}$

b) Find a basis for nullsp(A). Show your work, and $\boxed{\text{circle}}$ your final answer, which should be a list of vectors. *Hint: One of those vectors can involve b_1 and b_2.*

2

rank(A) + dim(nullsp(A)) = 5 → dim(nullsp(A)) = 3

need 3 LI vectors

① $\vec{x} = 3(\text{col } 1)$, so $A \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 3(\text{col } 1) - \vec{x} = \vec{0}$

② $\vec{y} = -1(\text{col } 1) + 2(\text{col } 2)$, so $A \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \vec{0}$

③ col 5 = 1(col 1) + 1(col 2), so $A \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \vec{0}$

$$\left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

basis for nullsp(A)

For the rest of the problem, let $\vec{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

c) Find **one** vector $\vec{w}'$ such that $A\vec{w}'$ is the orthogonal projection of $\vec{z}$ onto colsp(A). Show your work, and circle your final answer, which should be a vector with 5 entries.

colsp(A) = colsp(C), so let's find $\vec{w}'$ by using just C in the normal eq'ns.

$$\vec{w}' = (C^TC)^{-1}C^T\vec{z}$$

dot products of cols

$$C^TC = \begin{bmatrix} 15 & 0 \\ 0 & 3 \end{bmatrix}, \quad (C^TC)^{-1} = \begin{bmatrix} 1/15 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$\vec{w}' = (C^TC)^{-1}C^T\vec{z} = \begin{bmatrix} 1/15 & 0 \\ 0 & 1/3 \end{bmatrix}\underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{C^T\vec{z}} = \begin{bmatrix} 4/15 \\ 1/3 \end{bmatrix}$$

$C^T\vec{z}$: dot products of $\vec{z}$ with C's cols

→ Final answer:

$$\vec{w}' = \begin{bmatrix} 4/15 \\ 1/3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for the unused cols

d) Describe the **complete** set of vectors $\vec{w}^*$ such that $A\vec{w}^*$ is the orthogonal projection of $\vec{z}$ onto colsp(A). Show your work, and circle your final answer, which should be a set of vectors described in {set notation}.

$$\vec{w}^* = \left\{ \begin{bmatrix} 4/15 \\ 1/3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \; s, t, u \in \mathbb{R} \right\}$$

if you forgot this formula, notice that

$$\vec{p} = \left(\frac{\vec{z}\cdot\vec{v}}{\|\vec{z}\|^2}\right)\vec{z} = \left(\frac{v_1+v_2}{2}\right)\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}(v_1+v_2)/2\\(v_1+v_2)/2\\0\\0\\0\end{bmatrix} = P\begin{bmatrix}v_1\\v_2\\\vdots\\v_5\end{bmatrix}$$

e) Recall, $\vec{z} = \begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}$.

Find a matrix P such that $P\vec{v}$ is the orthogonal projection of $\vec{v}$ onto $\vec{z}$. Show your work, and $\boxed{\text{circle}}$ your final answer, which should be a matrix with no variables.

$$P = \frac{\vec{z}\vec{z}^T}{\underbrace{\vec{z}^T\vec{z}}_{}=2} \quad \text{from HW 5/7}$$

$$\vec{z}\vec{z}^T = \begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

$$P = \begin{bmatrix}1/2&1/2&0&0&0\\1/2&1/2&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

f) Find rank(PA), where P is the matrix you found in part **e)** and A is the matrix from the start of the problem.

○ 0 ● 1 ○ 2 ○ 3 ○ 4 ○ 5 ○ Impossible to tell

$$\text{rank}(PA) \leq \min(\text{rank}(P), \text{rank}(A))$$

here, $\text{rank}(P) = 1$ (outer product),

$\text{rank}(A) = 2$,

product is non-zero

so $\text{rank}(PA) > 0$,

so must be 1

$\text{nullsp}(A) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x - 3y + 4z = 0 \right\}$

Problem 4: The Null Space

Consider the plane in $\mathbb{R}^3$ given by the equation $x - 3y + 4z = 0$.

In parts **a)**, **b)**, and **c)**, let A be a matrix whose **null space** is the plane defined above.

Since nullsp is 2-d subsp of $\mathbb{R}^3$, A has 3 cols and rank $3-2=1$

a) Fill in the blanks:

rank$(A) = \boxed{1}$ dim(nullsp$(A)) = \boxed{2}$ (plane)

b) Suppose dim(nullsp$(A^T)) = 3$. Fill in the blanks:

A has $\boxed{4}$ rows and $\boxed{3}$ columns.

$\text{rank}(A^T) + \dim(\text{nullsp}(A^T)) = \#\text{rows} \rightarrow 1 + 3 = 4$

c) Find one possible matrix A where dim(nullsp$(A^T)) = 3$.

$$A = \begin{bmatrix} 1 & -3 & 4 \\ 1 & -3 & 4 \\ 1 & -3 & 4 \\ 1 & -3 & 4 \end{bmatrix}$$

if $A\vec{v} = \vec{0}$, then $\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \cdot \vec{v} = 0$ $\Rightarrow \vec{v}$ on plane

~~For the rest of the problem, let B be a matrix whose~~

d) Now, suppose B is a matrix whose **column space** is the plane defined above. Find one possible matrix B with 4 columns.

colsp is plane in $\mathbb{R}^3$: cols each in $\mathbb{R}^3$, rank 2, 4 cols

pick 2 LI vecs on plane: $\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$.

Then,
$$B = \begin{bmatrix} 4 & 4 & 0 & 0 \\ 0 & 0 & 4 & 4 \\ -1 & -1 & 3 & 3 \end{bmatrix}$$

Problem 5: Skincare Products

In this problem, we'll work with a dataset with information about skincare products. A sample of the dataset is shown below, but the full dataset has a total of n rows.

Price	Type	Brand	Sensitive	# Ingredients
55	Eye cream	PERRICONE MD	1	33
19	Cleanser	CLINIQUE	0	36
75	Eye cream	PETER THOMAS ROTH	1	42
38	Cleanser	PETER THOMAS ROTH	0	23
⋮	⋮	⋮	⋮	⋮

The `Sensitive` column contains either 1 or 0, corresponding to whether the product was designed for sensitive skin.

Our goal is to fit a multiple linear regression model (by minimizing mean squared error) that predicts **the number of ingredients** in a product given its `Price` and various other information.

a) Suppose we fit a model that uses an intercept term, `Price`, and `Sensitive` as features.

(i) Write the first two rows of the design matrix, X.

$$\begin{bmatrix} 1 & 55 & 1 \\ 1 & 19 & 0 \end{bmatrix}$$

(ii) Suppose that a solution to the normal equations is

$$\vec{w}^* = \begin{bmatrix} w_0^* \\ w_{\text{Price}}^* \\ w_{\text{Sensitive}}^* \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 8 \end{bmatrix}$$

This model is equivalent to **two** parallel lines in $\mathbb{R}^2$, each one of the form

$$h(\text{Price}_i) = a + b \cdot \text{Price}_i$$

where the line to use is determined by the value of `Sensitive`. Determine the values of a and b in both cases.

if Sensitive $=1$: $h(\text{Price}_i) = 5 - 2\cdot\text{Price}_i + 8$
$= 13 - 2\cdot\text{Price}_i \rightarrow$ $a=13$, $b=-2$

if sensitive$=0$: $a=5$, $b=-2$

The table below describes four possible models, each fit by minimizing mean squared error.

Model	Intercept?	Price?	Type?	Brand?	Sensitive?
Model 1	Yes	Yes	OHE, no categories dropped	No	No
Model 2	Yes	Yes	OHE, `"Eye cream"` dropped	No	No
Model 3	No	Yes	OHE, `"Eye cream"` dropped	OHE, no categories dropped	No
Model 4	No	Yes	OHE, `"Eye cream"` dropped	OHE, `"CLINIQUE"` dropped	No

In each part below, determine

- The number of columns in the design matrix X.
- The rank of the design matrix X.
- Whether or not the model's error vector $\vec{e}$ is guaranteed to sum to 0.

Assume that there are **8 unique values of** `Type` and **15 unique values of** `Brand`. **If it is impossible to determine the answer (e.g. if there are multiple possible answers), write "N/A"; don't just write one of them.**

$1+1+8$

one category redundant

b) Model 1

number of columns in X = $\boxed{10}$, rank(X) = $\boxed{9}$

Errors guaranteed to sum to 0? ● Yes ○ No

intercept term

c) Model 2 $1+1+8-1$

number of columns in X = $\boxed{9}$, rank(X) = $\boxed{9}$

Errors guaranteed to sum to 0? ● Yes ○ No

depends on relationship between Type and Brand cols!

d) Model 3 $0+1+(8-1)+15 = 23$

number of columns in X = $\boxed{23}$, rank(X) = $\boxed{\text{N/A}}$

Errors guaranteed to sum to 0? ● Yes ○ No

Brand categories sum to $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$.

e) Model 4

number of columns in X = $\boxed{22}$, rank(X) = $\boxed{\text{N/A}}$

Errors guaranteed to sum to 0? ○ Yes ● No

Problem 6: Gradients

Let

$$h(\vec{x}) = (C\vec{x}) \cdot (C\vec{x} + \vec{b})$$

where $\vec{x} \in \mathbb{R}^d$, C is an $n \times d$ matrix, and $\vec{b} \in \mathbb{R}^d$.

a) Find $\nabla h(\vec{x})$, the gradient of $h(\vec{x})$. $\boxed{\text{circle}}$ your final answer, which should be an expression in terms of $\vec{x}$, C, $\vec{b}$, and/or constants.

$$h(\vec{x}) = (C\vec{x})^T(C\vec{x}+\vec{b}) = \vec{x}^T C^T (C\vec{x}+\vec{b})$$

$$= \vec{x}^T \underbrace{C^T C}_{\text{quad form}} \vec{x} + \vec{x}^T \underbrace{C^T \vec{b}}_{\text{vector}}$$

$$\nabla h(\vec{x}) = 2C^T C\vec{x} + C^T\vec{b}$$

$$= 2C^T(C\vec{x}+\vec{b})$$

either form fine

b) Let

$$g(\vec{x}) = \log\left((h(\vec{x}))^2\right)$$

where log is the natural logarithm. Find $\nabla g(\vec{x})$, the gradient of $g(\vec{x})$. $\boxed{\text{circle}}$ your final answer, which should be an expression in terms of $\vec{x}$, C, $\vec{b}$, and/or constants.

Note $g(\vec{x}) = 2\log(h(\vec{x}))$ (easier to use)

Let $f(x) = \log(x)$ (scalar-to-scalar)

$$\nabla g(\vec{x}) = \frac{df}{dx}(h(\vec{x}))\,\nabla h(\vec{x})$$

$$= \frac{1}{h(\vec{x})}\, 2C^T(C\vec{x}+\vec{b}) = \frac{2C^T(C\vec{x}+\vec{b})}{(C\vec{x})^T(C\vec{x}+\vec{b})}$$

Problem 7: Gradient Descent

Suppose $\vec{u} \in \mathbb{R}^3$, and let

$$q(\vec{u}) = (u_1 + u_2 + u_3)^2 + (u_1 - u_2)^2 + (u_2 - u_3)^2$$

We write code that implements gradient descent in order to minimize q, using some initial guess, $\vec{u}^{(0)}$, and learning rate/step size, π. In our code, we add print statements that show us the values of $\vec{u}^{(t)}$ and $\nabla q(\vec{u}^{(t)})$ (the gradient of q) after each iteration.

Here's what we see:

After 1 iteration, $\vec{u}^{(1)} = \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}$, $\nabla q(\vec{u}^{(1)}) = \begin{bmatrix} -6 \\ -6 \\ -12 \end{bmatrix}$, $q(\vec{u}^{(1)}) = 21$

After 2 iterations, $\vec{u}^{(2)} = \begin{bmatrix} 1.2 \\ 0.2 \\ -0.6 \end{bmatrix}$, $\nabla q(\vec{u}^{(2)}) = \begin{bmatrix} 3.6 \\ 1.2 \\ 0 \end{bmatrix}$, $q(\vec{u}^{(2)}) = 2.28$

pick any component

$1.2 = 6\pi$

$\pi = \frac{1.2}{6} = 1/5$

a) What is value of π? Give your answer as a fraction or decimal.

$\pi =$ $\boxed{1/5}$

$\vec{u}^{(2)} = \vec{u}^{(1)} - \pi \nabla q(\vec{u}^{(1)})$

$$\begin{bmatrix} 1.2 \\ 0.2 \\ -0.6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} - \pi \begin{bmatrix} -6 \\ -6 \\ -12 \end{bmatrix}$$

b) What is the value of $\vec{u}_2^{(0)}$, i.e. what is the second component of the **initial guess vector**, $\vec{u}^{(0)}$? $\boxed{\text{Circle}}$ your final answer, which should be an expression in terms of π and/or constants. (If your answer does not involve π, we cannot give you partial credit in case your answer to (a) was incorrect.)

Need $\nabla q(\vec{u})$ to find $\nabla q(\vec{u}^{(0)})$, but only really care about 2nd component.

$$\nabla q(\vec{u}) = \begin{bmatrix} \cdot \\ \frac{\partial q}{\partial u_2} \\ \cdot \end{bmatrix}; \quad \frac{\partial q}{\partial u_2} = 2(u_1+u_2+u_3) + 2(u_1-u_2)(-1) + 2(u_2-u_3)$$

$$= 6u_2$$

so, $\vec{u}^{(1)} = \vec{u}^{(0)} - \pi \nabla q(\vec{u}^{(0)})$, or

$u_2^{(1)} = u_2^{(0)} - \pi(6u_2^{(0)}) \Rightarrow -1 = u_2^{(0)}(1-6\pi)$

$\Rightarrow u_2^{(0)} = \frac{-1}{1-6\pi} = 5$

Problem 8: Convexity

In Lab 10, we proved that if $f : \mathbb{R} \to \mathbb{R}$ is a convex function such that $f(0) = 0$, then for all $y \in \mathbb{R}$ and $t \in [0, 1]$,

$$f(ty) \leq tf(y)$$

It turns out that this fact is true for all real numbers t, not just $t \in [0, 1]$ (as long as f is convex and $f(0) = 0$).

Using this fact, prove that if $f : \mathbb{R} \to \mathbb{R}$ is a convex function such that $f(0) = 0$, then for all $x, y \in \mathbb{R}$,

$$f(x + y) \leq f(x) + f(y)$$

Hint: Start by using the definition of convexity on f, x, and y, using $t = \frac{1}{2}$. This will tell you something about $f(x) + f(y)$. To relate this to $f(x + y)$, use the first fact provided from Lab 10, but substitute $t = 2$ and something involving both x and y for y.

Using the hint, since f is convex:

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$$

at $t = 1/2$

$$f(1/2\, x + 1/2\, y) \leq 1/2\, f(x) + 1/2\, f(y)$$

$$2f\left(\frac{x+y}{2}\right) \leq \underbrace{f(x) + f(y)}_{\text{upper bound}}$$

Now, let $z = \dfrac{x+y}{2}$. For all t, since $f(0) = 0$ (and f convex),

$$f(tz) \leq tf(z) \qquad \text{pick } t = 2$$

$$f(2z) \leq 2f(z)$$

$$\underbrace{f(x+y)}_{\text{lower bound}} \leq 2f\left(\frac{x+y}{2}\right) \leq \underbrace{f(x) + f(y)}_{\text{from earlier}}$$

So, $f(x+y) \leq f(x) + f(y)$.

Congrats on finishing Mock Midterm 2!

Feel free to draw us a picture about EECS 245 in the box below.