Mock Midterm 2

EECS 245, Fall 2025 at the University of Michigan

Name: _____

uniqname: _____

UMID: _____

Room: ◯ 1365 LCSIB   ◯ 2901 BBB

Instructions

- This exam consists of 8 problems, worth a total of (N/A) points, spread across 14 pages (7 sheets of paper).

- You have 80 minutes to complete this exam, unless you have extended-time accommodations through SSD.

- Write your uniqname in the top right corner of each page.

- For free response problems, you must show all of your work (unless otherwise specified), and ⬚circle⬚ your final answer. We will not grade work that appears elsewhere, and you may lose points if your work is not shown.

- For multiple choice problems, completely fill in bubbles and square boxes; if we cannot tell which option(s) you selected, you may lose points.
  ◯ A bubble means that you should only select one choice.
  ☐ A square box means you should select all that apply.

- You may refer to a single two-sided handwritten notes sheet. Other than that, you may not refer to any other resources or technology during the exam (no phones, watches, or calculators).

You are to abide by the University of Michigan/Engineering Honor Code. To receive a grade, please sign below to signify that you have kept the Honor Code pledge.

I have neither given nor received aid on this exam, nor have I concealed any violations of the Honor Code.

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Problem 1: Inversion

Let $A = \begin{bmatrix} 2 & -3 \\ 8 & -10 \end{bmatrix}$.

a) Find $\det(A)$, the determinant of A.

$\det(A) =$

b) Find the area of the four-sided polygon in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,4)$, $(-2,-6)$, and $(-3,-10)$.

Area $=$

c) Find A^{-1}, the inverse of A.

$A^{-1} =$

Problem 2: True or False?

Let A be an $n \times d$ matrix. For each statement in this problem, select True or False. For True statements, provide a short justification; for False statements, provide a counterexample (or a general argument for why the statement is false).

a) If A is square, and $\vec{x} \in \text{nullsp}(A)$ and $\vec{y} \in \text{colsp}(A)$, then $\vec{x} \cdot \vec{y} = 0$.

  ◯ True   ◯ False

b) If A is square, and $\vec{x} \in \text{nullsp}(A)$ and $\vec{y} \in \text{colsp}(A^T)$, then $\vec{x} \cdot \vec{y} = 0$.

  ◯ True   ◯ False

c) If A is symmetric $(A = A^T)$, and $\vec{x} \in \text{nullsp}(A)$ and $\vec{y} \in \text{colsp}(A)$, then $\vec{x} \cdot \vec{y} = 0$.

  ◯ True   ◯ False

d) If P projects vectors in $\mathbb{R}^n$ onto the column space of A, and $\vec{x}, \vec{y} \in \mathbb{R}^n$ are orthogonal, then $P\vec{x}$ and $P\vec{y}$ are also orthogonal.

  ◯ True   ◯ False

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Problem 3: CR decomposition

Suppose $\vec{x}, \vec{y} \in \mathbb{R}^5$ and let

$$A = \begin{bmatrix} 3 & 0 & | & | & 3 \\ 1 & 1 & | & | & 2 \\ -1 & 1 & \vec{x} & \vec{y} & 0 \\ 0 & 1 & | & | & 1 \\ 2 & 0 & | & | & 2 \end{bmatrix}$$

Additionally, suppose A has the following CR decomposition:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ -1 & 1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & -1 & b_1 \\ 0 & 1 & 0 & 2 & b_2 \end{bmatrix}$$

a) Determine the following values.

$\text{rank}(A) = $ [   ] , $b_1 = $ [   ] , $b_2 = $ [   ]

b) Find a basis for $\text{nullsp}(A)$. Show your work, and [circle] your final answer, which should be a list of vectors. *Hint: One of those vectors can involve b_1 and b_2.*

For the rest of the problem, let $\vec{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

c) Find **one** vector $\vec{w}'$ such that $A\vec{w}'$ is the orthogonal projection of $\vec{z}$ onto $\mathrm{colsp}(A)$. Show your work, and [circle] your final answer, which should be a vector with 5 entries.

d) Describe the **complete** set of vectors $\vec{w}^*$ such that $A\vec{w}^*$ is the orthogonal projection of $\vec{z}$ onto $\mathrm{colsp}(A)$. Show your work, and [circle] your final answer, which should be a set of vectors described in {set notation}.

e) Recall, $\vec{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Find a matrix P such that $P\vec{v}$ is the orthogonal projection of $\vec{v}$ onto $\vec{z}$. Show your work, and
circle your final answer, which should be a matrix with no variables.

f) Find $\text{rank}(PA)$, where P is the matrix you found in part **e)** and A is the matrix from the start
of the problem.

○ 0   ○ 1   ○ 2   ○ 3   ○ 4   ○ 5   ○ Impossible to tell

Problem 4: The Null Space

Consider the plane in $\mathbb{R}^3$ given by the equation $x - 3y + 4z = 0$.

In parts **a)**, **b)**, and **c)**, let A be a matrix whose **null space** is the plane defined above.

a) Fill in the blanks:

$\text{rank}(A) =$ [   ] $\dim(\text{nullsp}(A)) =$ [   ]

b) Suppose $\dim(\text{nullsp}(A^T)) = 3$. Fill in the blanks:

A has [   ] rows and [   ] columns.

c) Find one possible matrix A where $\dim(\text{nullsp}(A^T)) = 3$.

For the rest of the problem, let B be a matrix whose

d) Now, suppose B is a matrix whose **column space** is the plane defined above. Find one possible matrix B with 4 columns.

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Problem 5: Skincare Products

In this problem, we'll work with a dataset with information about skincare products. A sample of the dataset is shown below, but the full dataset has a total of n rows.

Price	Type	Brand	Sensitive	# Ingredients
55	Eye cream	PERRICONE MD	1	33
19	Cleanser	CLINIQUE	0	36
75	Eye cream	PETER THOMAS ROTH	1	42
38	Cleanser	PETER THOMAS ROTH	0	23
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The `Sensitive` column contains either 1 or 0, corresponding to whether the product was designed for sensitive skin.

Our goal is to fit a multiple linear regression model (by minimizing mean squared error) that predicts **the number of ingredients** in a product given its `Price` and various other information.

a) Suppose we fit a model that uses an intercept term, `Price`, and `Sensitive` as features.

(i) Write the first two rows of the design matrix, X.

(ii) Suppose that a solution to the normal equations is

$$\vec{w}^* = \begin{bmatrix} w_0^* \\ w_{\text{Price}}^* \\ w_{\text{Sensitive}}^* \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 8 \end{bmatrix}$$

This model is equivalent to **two** parallel lines in $\mathbb{R}^2$, each one of the form

$$h(\text{Price}_i) = a + b \cdot \text{Price}_i$$

where the line to use is determined by the value of `Sensitive`. Determine the values of a and b in both cases.

The table below describes four possible models, each fit by minimizing mean squared error.

Model	Intercept?	Price?	Type?	Brand?	Sensitive?
Model 1	Yes	Yes	OHE, no categories dropped	No	No
Model 2	Yes	Yes	OHE, "Eye cream" dropped	No	No
Model 3	No	Yes	OHE, "Eye cream" dropped	OHE, no categories dropped	No
Model 4	No	Yes	OHE, "Eye cream" dropped	OHE, "CLINIQUE" dropped	No

In each part below, determine

- The number of columns in the design matrix X.

- The rank of the design matrix X.

- Whether or not the model's error vector $\vec{e}$ is guaranteed to sum to 0.

Assume that there are **8 unique values of** `Type` and **15 unique values of** `Brand`. **If it is impossible to determine the answer (e.g. if there are multiple possible answers), write "N/A"; don't just write one of them.**

b) **Model 1**

number of columns in $X = $ [____], rank$(X) = $ [____]

Errors guaranteed to sum to 0? ◯ Yes ◯ No

c) **Model 2**

number of columns in $X = $ [____], rank$(X) = $ [____]

Errors guaranteed to sum to 0? ◯ Yes ◯ No

d) **Model 3**

number of columns in $X = $ [____], rank$(X) = $ [____]

Errors guaranteed to sum to 0? ◯ Yes ◯ No

e) **Model 4**

number of columns in $X = $ [____], rank$(X) = $ [____]

Errors guaranteed to sum to 0? ◯ Yes ◯ No

Problem 6: Gradients

Let

$$h(\vec{x}) = (C\vec{x}) \cdot (C\vec{x} + \vec{b})$$

where $\vec{x} \in \mathbb{R}^d$, C is an $n \times d$ matrix, and $\vec{b} \in \mathbb{R}^d$.

a) Find $\nabla h(\vec{x})$, the gradient of $h(\vec{x})$. $\boxed{\text{circle}}$ your final answer, which should be an expression in terms of $\vec{x}$, C, $\vec{b}$, and/or constants.

b) Let

$$g(\vec{x}) = \log\left((h(\vec{x}))^2\right)$$

where $\log$ is the natural logarithm. Find $\nabla g(\vec{x})$, the gradient of $g(\vec{x})$. $\boxed{\text{circle}}$ your final answer, which should be an expression in terms of $\vec{x}$, C, $\vec{b}$, and/or constants.

Problem 7: Gradient Descent

Suppose $\vec{u} \in \mathbb{R}^3$, and let

$$q(\vec{u}) = (u_1 + u_2 + u_3)^2 + (u_1 - u_2)^2 + (u_2 - u_3)^2$$

We write code that implements gradient descent in order to minimize q, using some initial guess, $\vec{u}^{(0)}$, and learning rate/step size, π. In our code, we add print statements that show us the values of $\vec{u}^{(t)}$ and $\nabla q(\vec{u}^{(t)})$ (the gradient of q) after each iteration.

Here's what we see:

$$\text{After 1 iteration, } \vec{u}^{(1)} = \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}, \nabla q(\vec{u}^{(1)}) = \begin{bmatrix} -6 \\ -6 \\ -12 \end{bmatrix}, q(\vec{u}^{(1)}) = 21$$

$$\text{After 2 iterations, } \vec{u}^{(2)} = \begin{bmatrix} 1.2 \\ 0.2 \\ -0.6 \end{bmatrix}, \nabla q(\vec{u}^{(2)}) = \begin{bmatrix} 3.6 \\ 1.2 \\ 0 \end{bmatrix}, q(\vec{u}^{(2)}) = 2.28$$

a) What is value of π? Give your answer as a fraction or decimal.

$\pi =$

b) What is the value of $\vec{u}_2^{(0)}$, i.e. what is the second component of the **initial guess vector**, $\vec{u}^{(0)}$? $\boxed{\text{Circle}}$ your final answer, which should be an expression in terms of π and/or constants. (If your answer does not involve π, we cannot give you partial credit in case your answer to (a) was incorrect.)

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Problem 8: Convexity

In Lab 10, we proved that if $f : \mathbb{R} \to \mathbb{R}$ is a convex function such that $f(0) = 0$, then for all $y \in \mathbb{R}$ and $t \in [0, 1]$,

$$f(ty) \leq tf(y)$$

It turns out that this fact is true for all real numbers t, not just $t \in [0, 1]$ (as long as f is convex and $f(0) = 0$).
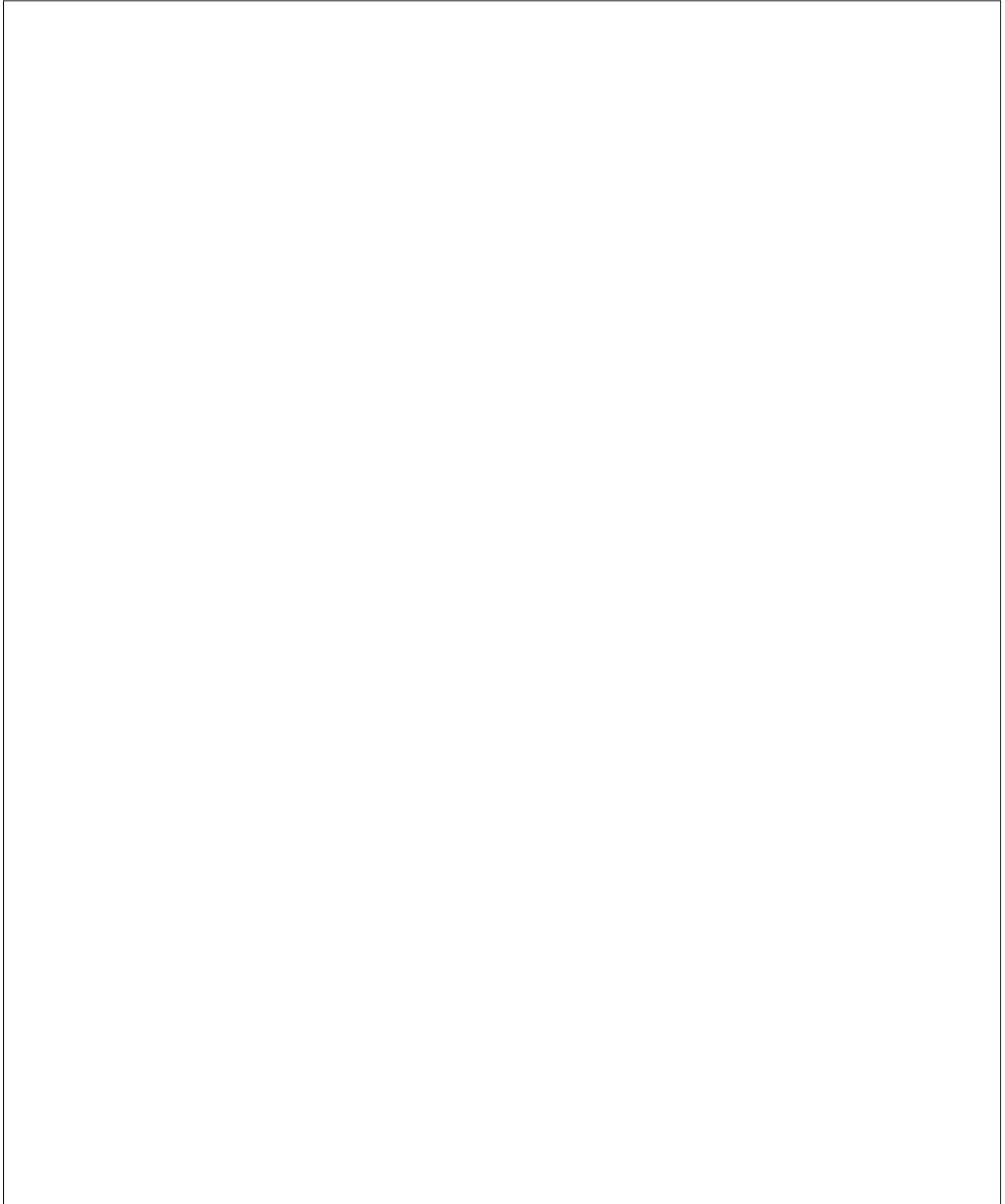
Using this fact, prove that if $f : \mathbb{R} \to \mathbb{R}$ is a convex function such that $f(0) = 0$, then for all $x, y \in \mathbb{R}$,

$$f(x + y) \leq f(x) + f(y)$$

Hint: Start by using the definition of convexity on f, x, and y, using $t = \frac{1}{2}$. This will tell you something about $f(x) + f(y)$. To relate this to $f(x + y)$, use the first fact provided from Lab 10, but substitute $t = 2$ and something involving both x and y for y.

Congrats on finishing Mock Midterm 2!

Feel free to draw us a picture about EECS 245 in the box below.