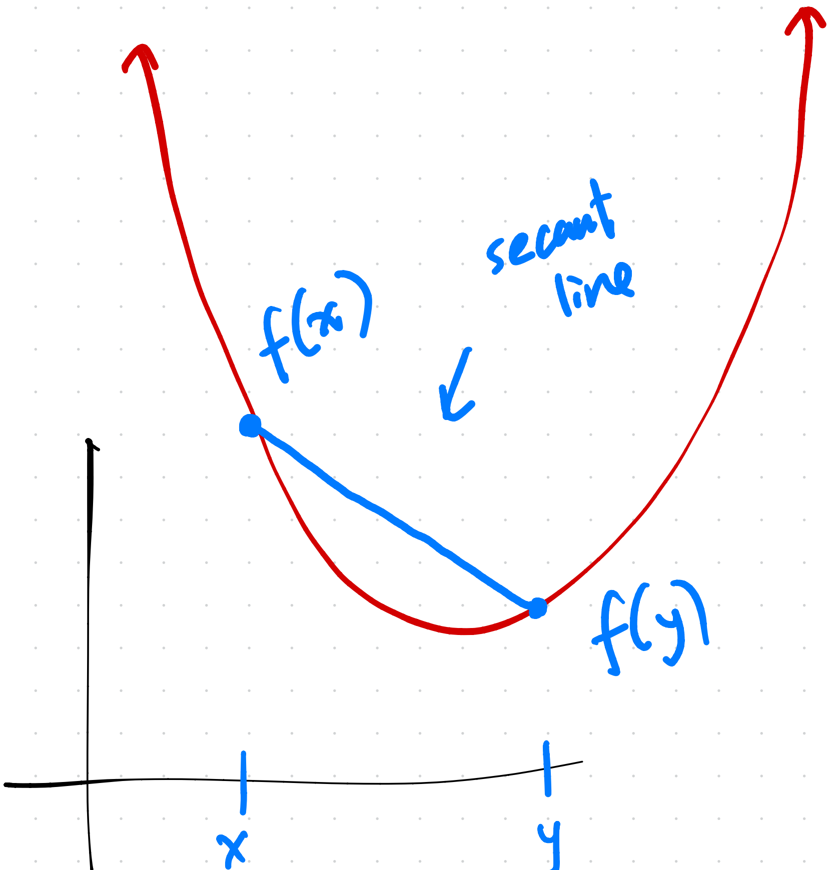
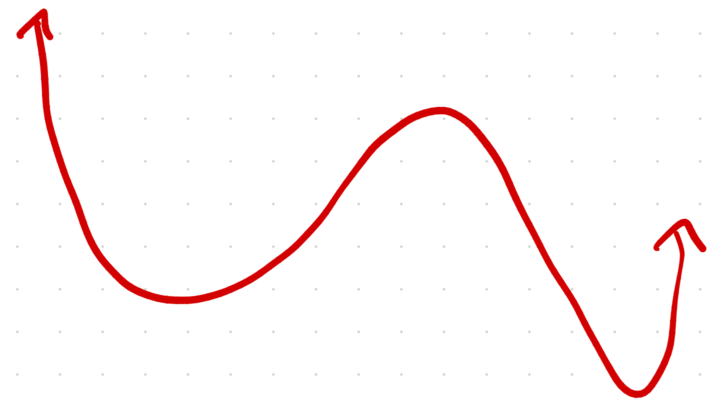
EECS 245 Fall 2025

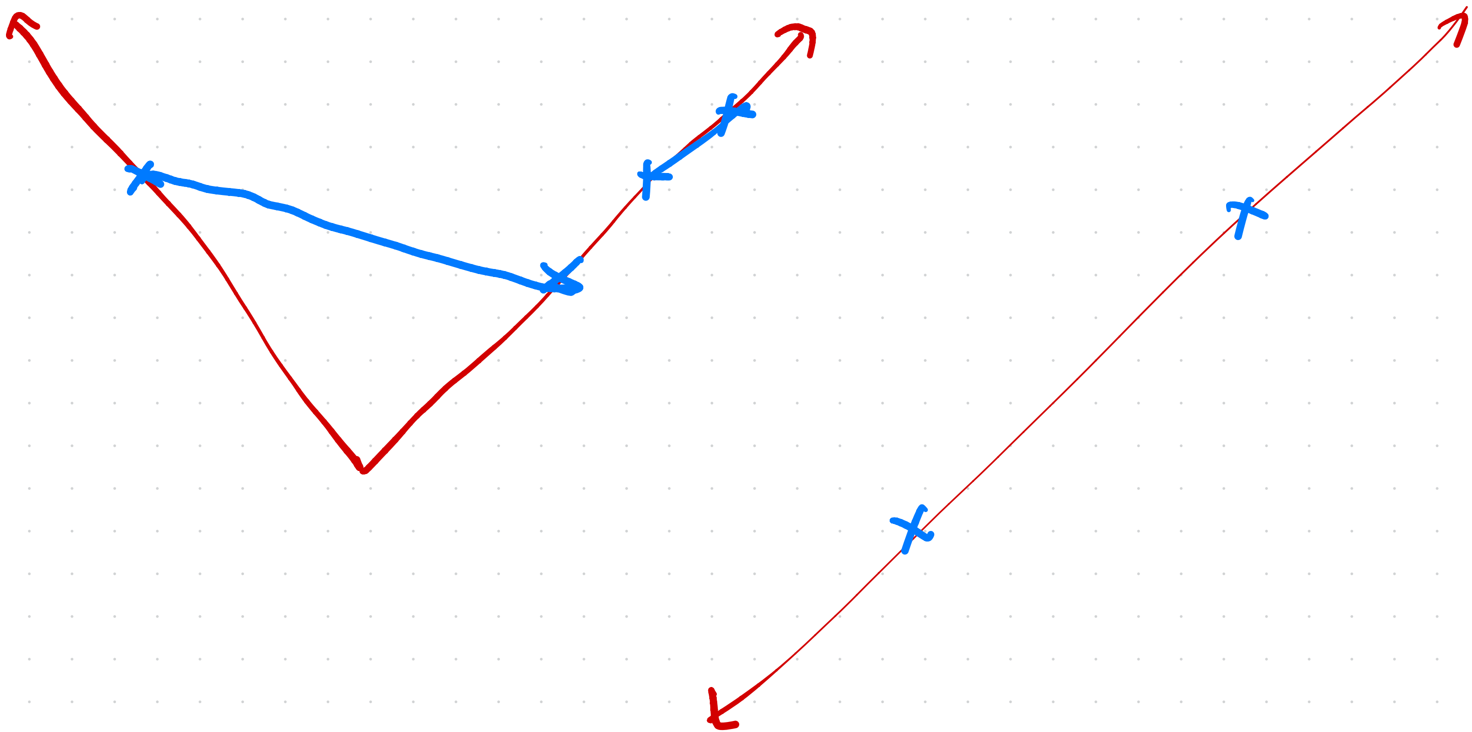
Math for ML

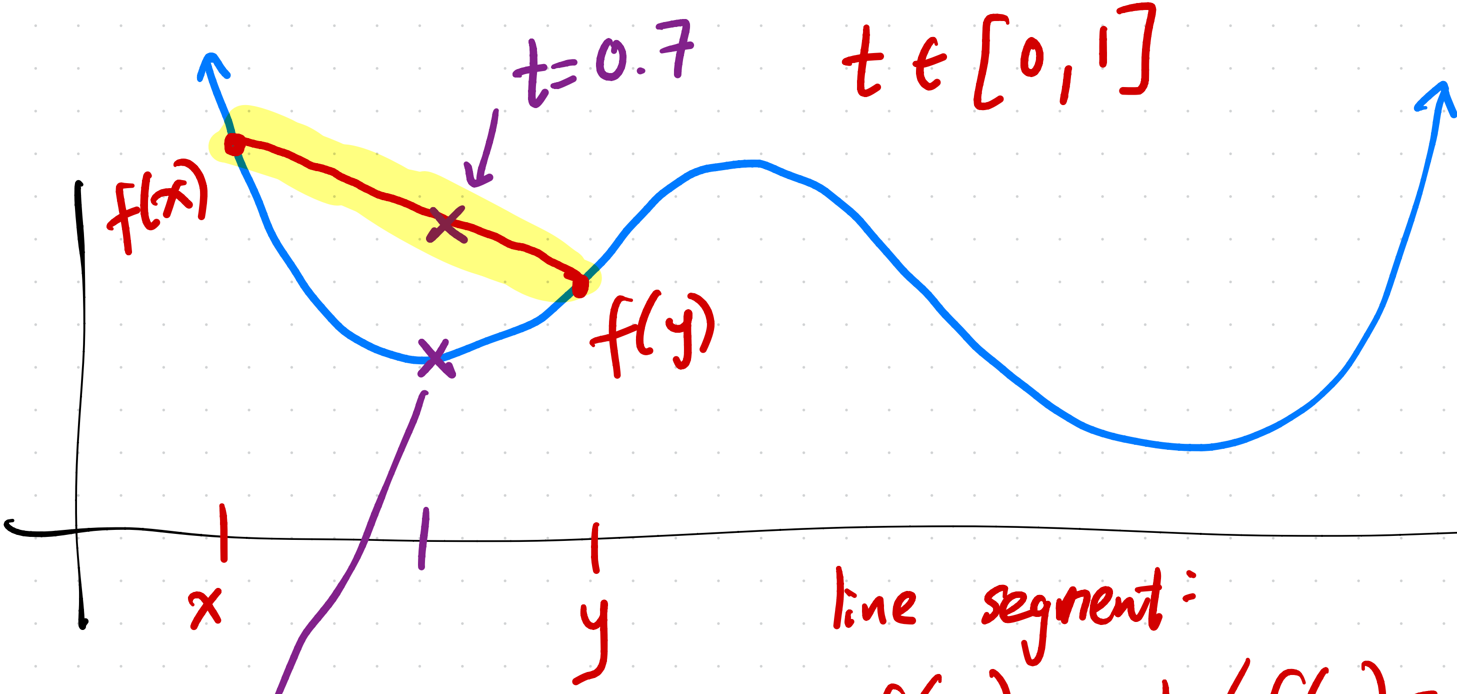
Lecture 19: Convexity

Convexity









$$f(\underbrace{(1-t)x + ty})$$

line segment: $t \in [0,1]$

$$L: \quad f(x) + t\left(f(y) - f(x)\right)$$
$$= f(x) + t f(y) - t f(x)$$
$$= (1-t) f(x) + t f(y)$$

Convexity

$f : \mathbb{R}^d \to \mathbb{R}$ is convex if

for all $\vec{x}, \vec{y}$ in $\mathbb{R}^d$ and

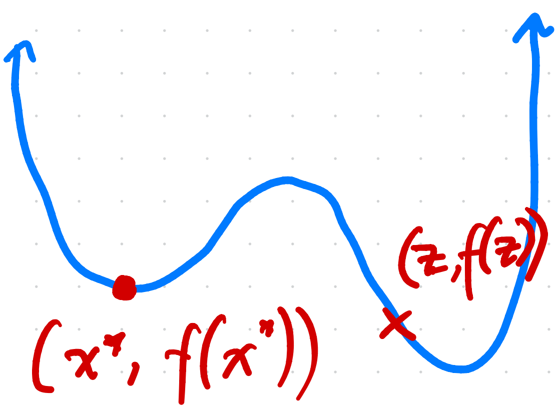
all $0 \le t \le 1$,

$$f((1-t)\vec{x} + t\vec{y}) \le (1-t)f(\vec{x}) + t f(\vec{y})$$

$f \quad \le \quad$ line segment

if $f : \mathbb{R}^d \to \mathbb{R}$ is convex, then any local min is a global min

<u>contradiction</u> : assume $\vec{x}^*$ is a local min, but $f(\vec{z}) < f(\vec{x}^*)$



formal def

$$f((1-t)\vec{x}^* + t\vec{z}) \le (1-t)f(\vec{x}^*) + t f(\vec{z})$$

$$< (1-t) f(\vec{x}^*) + t f(\vec{x}^*)$$

$$= f(\vec{x}^*) - \cancel{t f(\vec{x}^*)} + \cancel{t f(\vec{x}^*)}$$

$$f\left((1-t)\vec{x}^* + t\vec{z}\right) < f(\vec{x}^*)$$

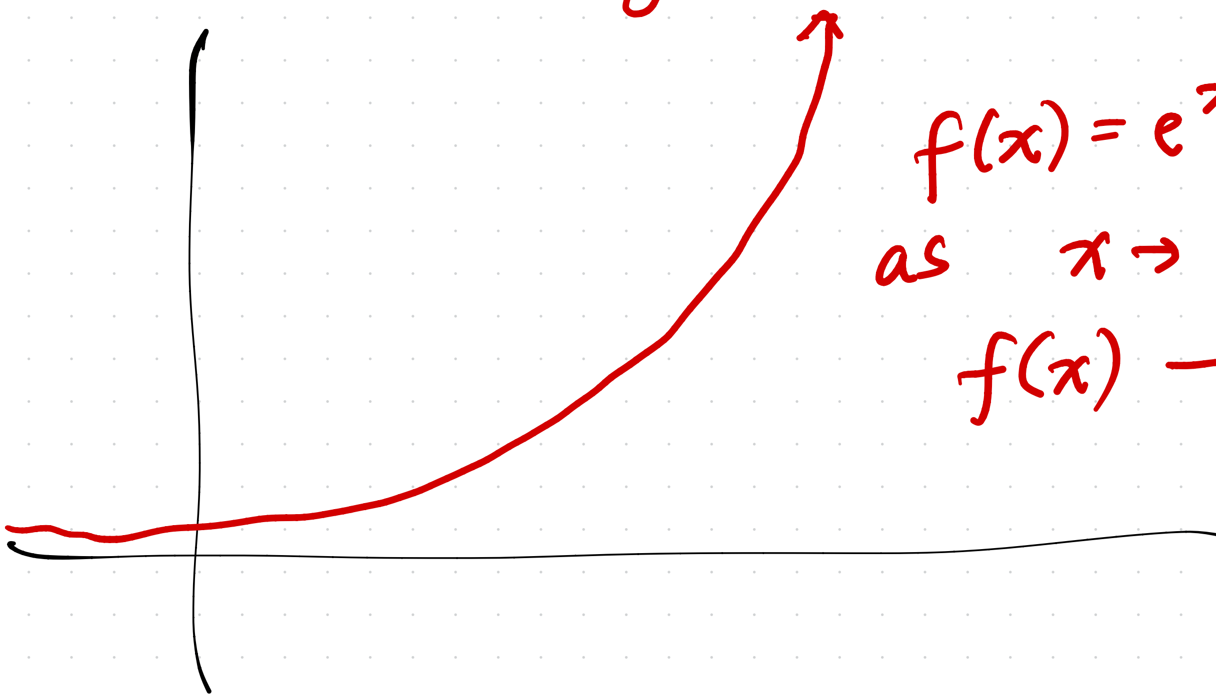
why contradiction ?

$t=0$

$$\underbrace{f(\vec{x}^*) < f(\vec{x}^*)}$$

contradiction !

do all convex functions have a global min?



$f(x) = e^x$

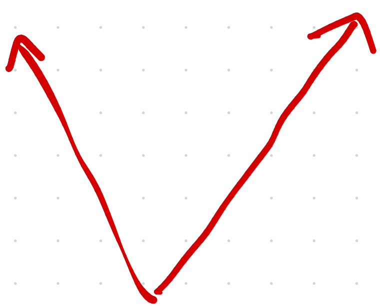
as $x \to -\infty$,

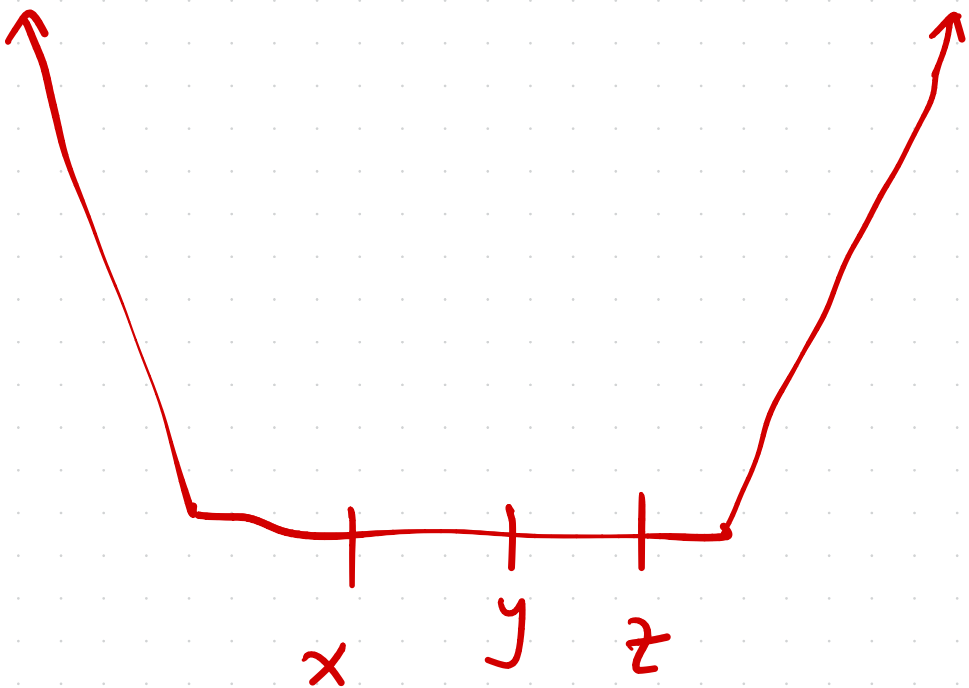
$f(x) \to 0$

strict convexity

for all $\vec{x}, \vec{y} \in \mathbb{R}^d$,
$t \in [0,1]$

$$f\big((1-t)\vec{x} + t\vec{y}\big) < (1-t)f(\vec{x}) + t f(\vec{y})$$





convex

local min = global min

BuT multiple minimizers

if f strictly convex, then its global min (if exists) is unique ✓

contradiction

suppose $\vec{x}^* \neq \vec{y}^*$ different, but both global minimizers of f, i.e.

$$f(\vec{x}^*) = f(\vec{y}^*) = m$$

$$\begin{aligned} f\left((1-t)\vec{x}^* + t\vec{y}^*\right) &< (1-t) f(\vec{x}^*) + t f(\vec{y}^*) \\ &= (1-t)m + tm \\ &= m \end{aligned}$$

$\rightarrow$ contradiction!

$$R_{sq}(\vec{w}) = \frac{1}{n}\|\vec{y} - X\vec{w}\|^2$$

always convex

can you think of an example where R is <u>not</u> strictly convex (but is convex)?