EECS 245 Fall 2025
Math for ML

Lecture 20: Review Convexity,
start Eigenvalues and Eigenvectors

Agenda

Tuesday's recording!

→ Recap: Convexity

→ what are eigenvalues and eigenvectors?

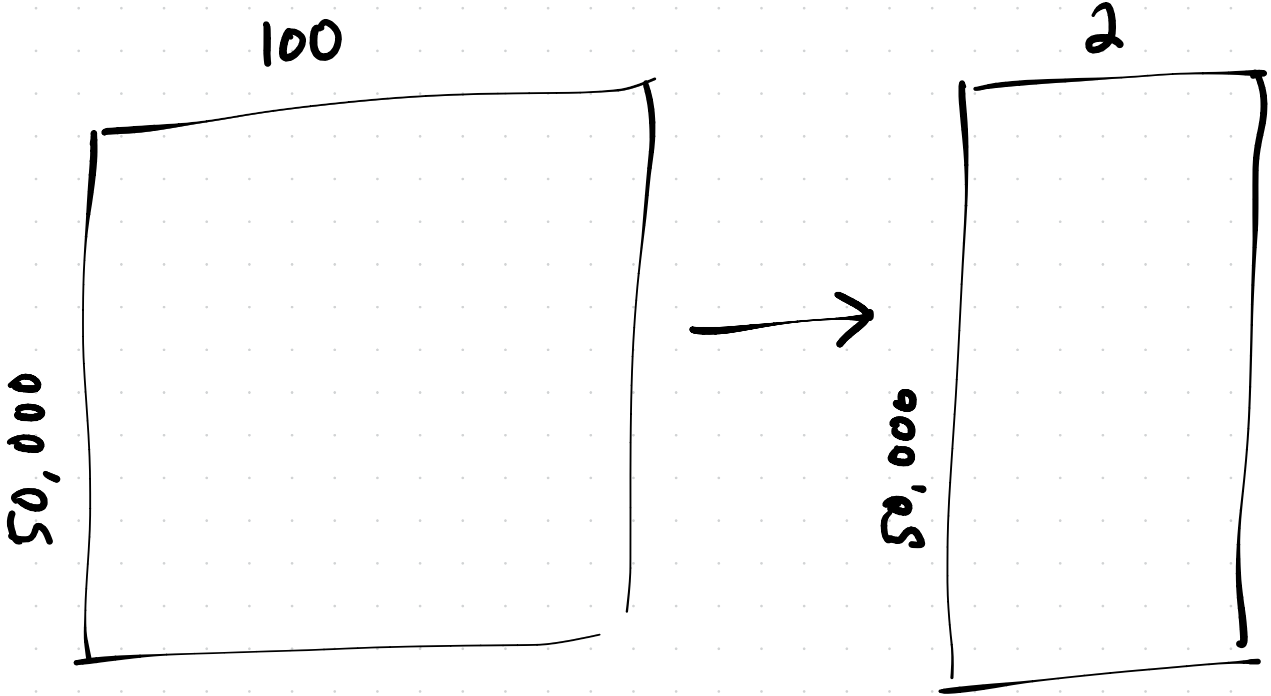
→ application: Google's PageRank algorithm

eventually will be chapter 5, but not on MT 2

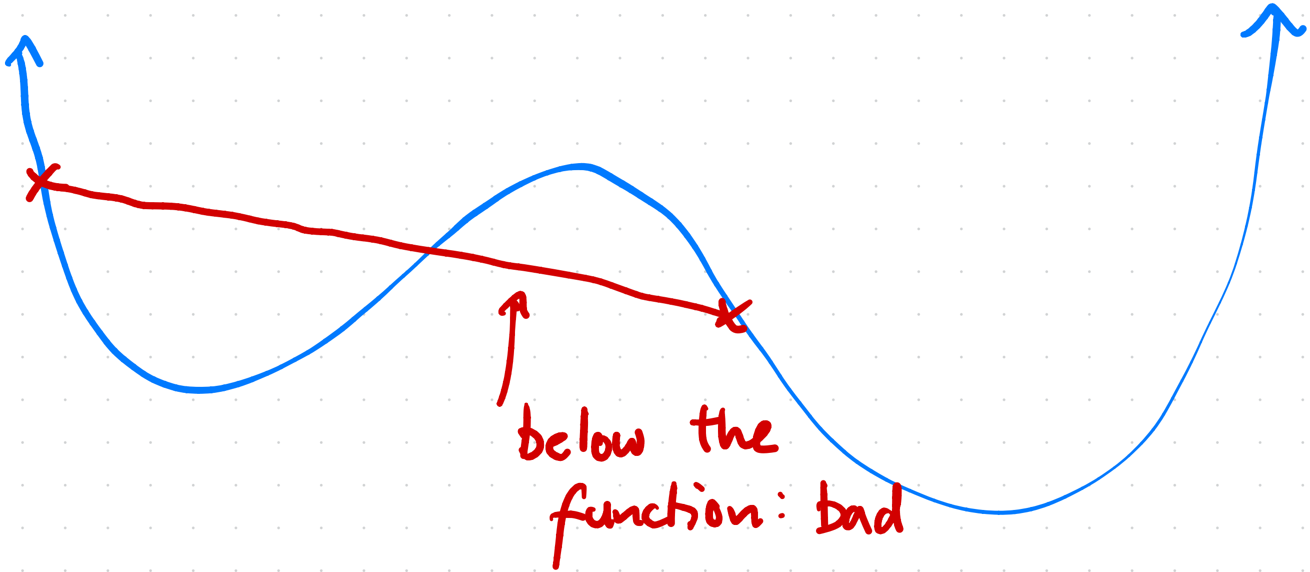
Announcements

1. HW 9 due tomorrow
2. Mock exam tomorrow

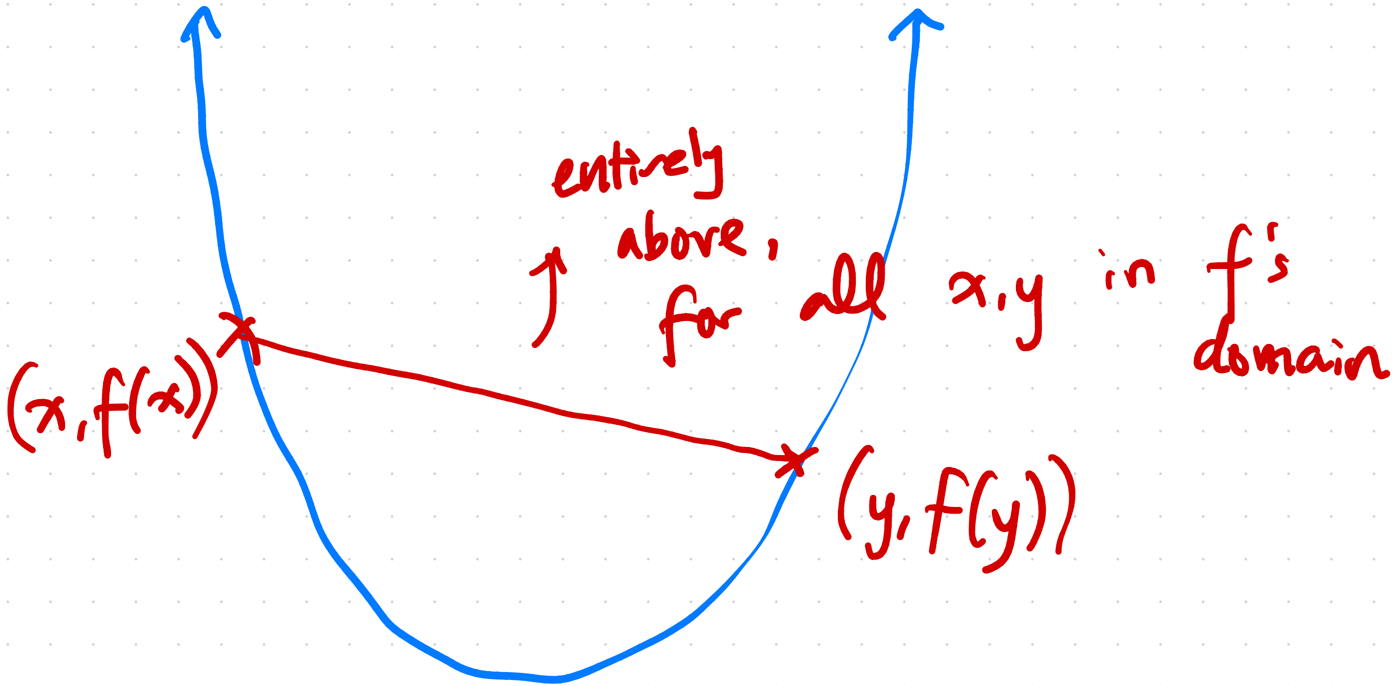
   2:30 – 5:30 PM, 1365 LCSIB
3. New Resources tab on the course website!
4. Midterm 2 in lecture on Tuesday: see Ed for detail

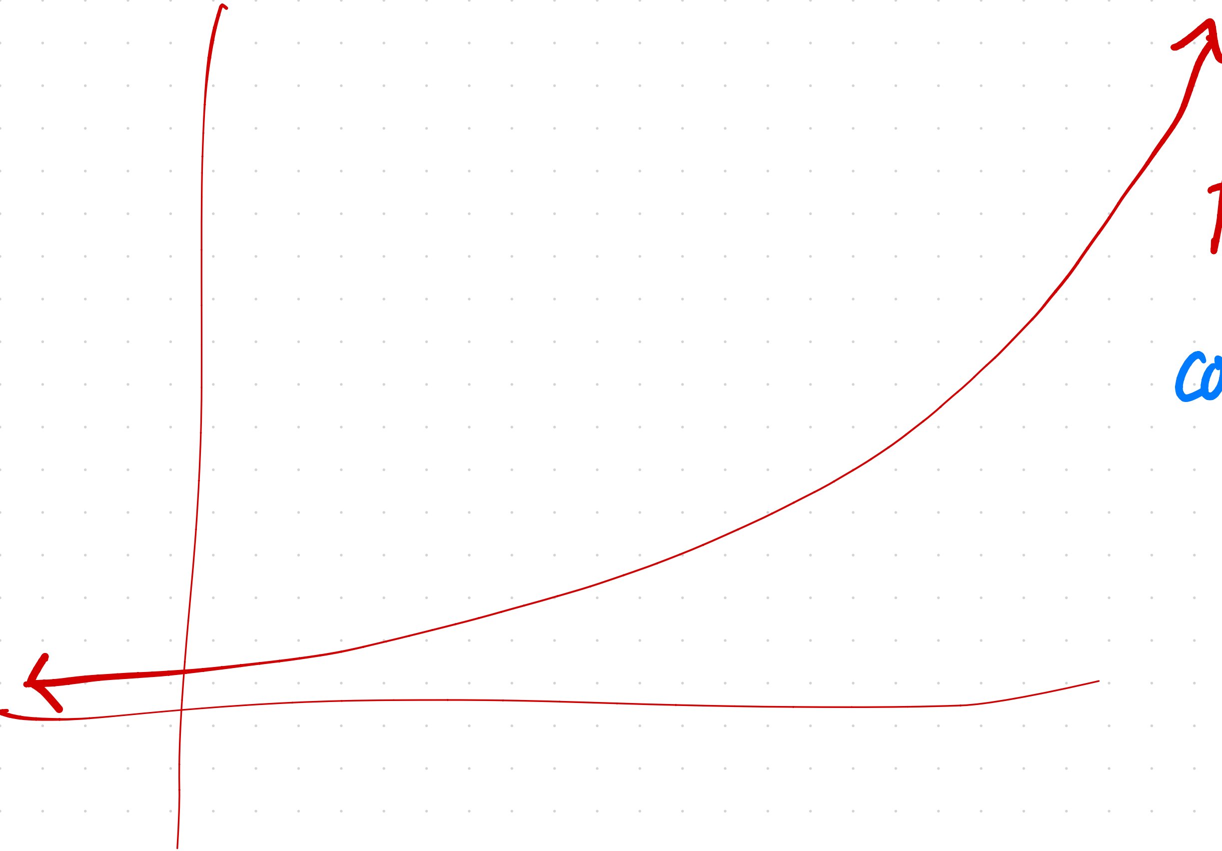

100
50,000
2
50,000

Convexity: secant line between any 2 points is always on or above f



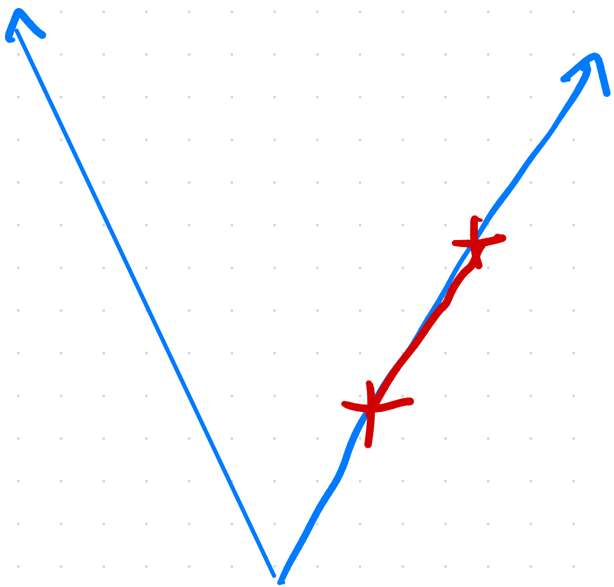
not convex!


entirely
above,
for all x, y in f's
domain
(x, f(x))
(y, f(y))



$f(x) = e^x$

convex, but
no minimum!



convex, but <u>not</u>

strictly convex

$$f\big(\underbrace{(1-t)\vec{x} + t\vec{y}}_{\text{function}}\big) \underset{\geq}{\leq} \underbrace{(1-t)f(\vec{x}) + t f(\vec{y})}_{\text{line segment}}$$

for all $\vec{x}, \vec{y}$ in domain of f
and all $t \in [0,1]$

strict:
$\vec{x} \neq \vec{y}$, $<$ instead of $\leq$,

$\underbrace{t \in (0,1)}_{\text{don't allow } 0,1}$ instead of $\underbrace{[0,1]}_{\text{do allow}}$

$$R_{sq}(\vec{w}) = \frac{1}{n}\|\vec{y} - X\vec{w}\|^2$$

convex ✓

for scalar-to-scalar functions,

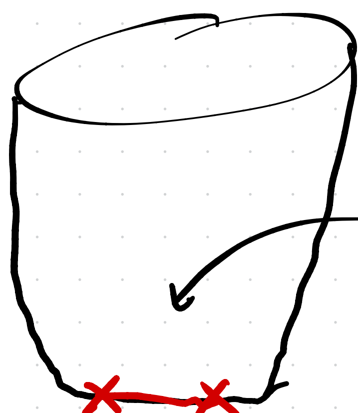
second derivative test says if

$$\frac{d^2 f}{dx^2} > 0$$ for all x in f's domain,

then f is convex

$$R_{sq}(\vec{w}) = \frac{1}{n} \|\vec{y} - X\vec{w}\|^2$$

if X's cols are <u>not</u> linearly independent,
then X^TX is not invertible,
and infinitely many $\vec{w}^*$'s minimize R.



flat valley at bottom
not strictly convex

for vector-to-scalar functions, there are many 2^{nd} derivatives

$$f(\vec{x}) = x_1^2 + 3x_1x_2 + 2x_2^2$$

$$\nabla f(\vec{x}) = \begin{array}{cc} \begin{bmatrix} 2x_1 + 3x_2 \\ 3x_1 + 4x_2 \end{bmatrix} & \begin{array}{c} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{array} \end{array}$$

$$\nabla^2 f(\vec{x}) = H = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

"Hessian"

$$\overbrace{\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}}^{H}$$

second-derivative test for vector-to-scalar f:

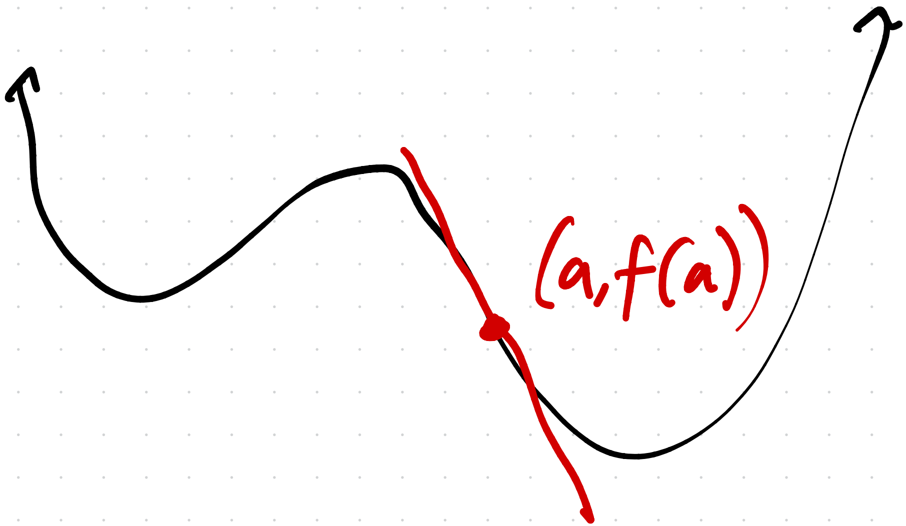
$H = \nabla^2 f(\vec{x})$ = matrix of second derivatives,

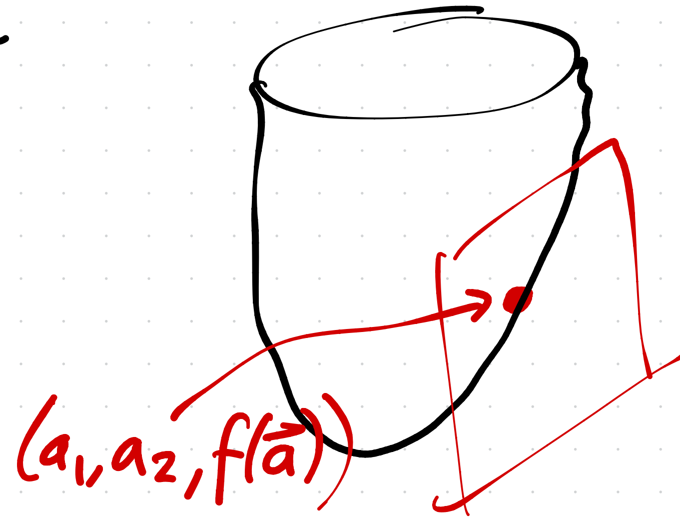
then f convex iff

$$\vec{x}^T H \vec{x} \geq 0 \quad \text{for all } \vec{x} \in \mathbb{R}^d$$

"H is positive semi-definite"

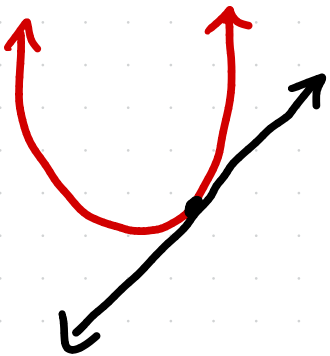
Tangent hyperplane





$$f(\vec{x}) \approx f(\vec{a}) + \left(\nabla f(\vec{a})\right)^T (\vec{x} - \vec{a})$$

If f is differentiable, then at any point



$\vec{a}$, the tangent hyperplane always lies <u>below</u> the function

if strict: $>$

$$f(\vec{y}) \geq f(\vec{x}) + \left(\nabla f(\vec{x})\right)^T (\vec{y} - \vec{x})$$

function above tangent hyperplane

for all $\vec{x}, \vec{y}$ in domain of f, $\vec{y} \neq \vec{x}$

Eigenvalues and eigenvectors

suppose A is a square, $n \times n$ matrix.

λ is an eigenvalue of A, if there exists some $\vec{v} \in \mathbb{R}^n$ such that

$$A\vec{v} = \lambda \vec{v}$$

$\vec{v}$ is the corresponding eigenvector $(\vec{v} \neq \vec{0})$

example:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix} k,$$

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ not an eigenvector of A

$$A \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 5 \\ 0 \end{bmatrix},$$

so $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ is an eigenvector of A, $\lambda = 2$

in fact, any scalar multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector, with $\lambda = 2$!

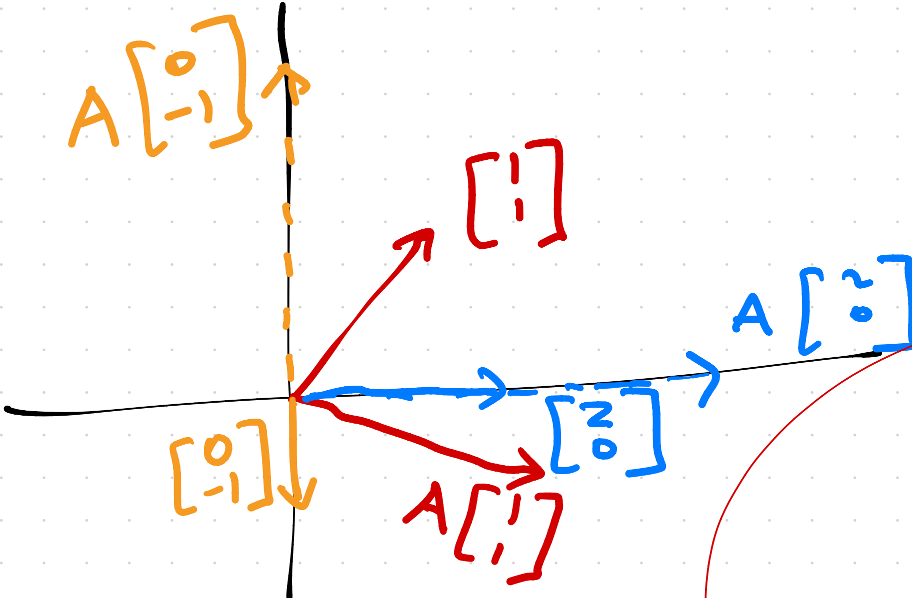
$$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & -3\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-3\end{bmatrix} = -3\begin{bmatrix}0\\1\end{bmatrix},$$

so $\lambda = -3$ is also an eigenvalue;

any scalar multiple of $\begin{bmatrix}0\\1\end{bmatrix}$

is the corresponding eigenvector

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$



intuition:

A's eigenvectors preserve direction when multiplied by A e.g.

$$Q = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$ orthogonal

Q is a rotation matrix,

so no vector $\vec{v}$ will point in the same direction as $Q\vec{v}$

$\Rightarrow$ no real-valued eigenvalues/eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

identity matrix

$\vec{v}$ is in $A-\lambda I$'s null space

$\Rightarrow$ $A-\lambda I$ has a non-trivial null space

$\Rightarrow$ $\det(A-\lambda I) = 0$ $\rightarrow$ use this to find λ's!

"characteristic polynomial" of A:

$$p(\lambda) = \det(A - \lambda I)$$

zeros of $p(\lambda)$ → eigenvalues, λ

$$\det(A - \lambda I) = 0$$

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \quad \underset{A-\lambda I}{\begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}}$$

$$\det(A-\lambda I) = 0$$

$$(1-\lambda)(4-\lambda) - 6 = 0$$

$$4 - 5\lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 - 5\lambda - 2 = 0$$

quadratic eq'n

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$p(\lambda) = (1-\lambda)^2 - 4 = 0$$

$$1 - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda+1)(\lambda-3) = 0$$

what are the corresponding $\vec{v}$'s?

$\lambda = -1$,

$\lambda = 3$