Final Exam

EECS 245, Fall 2025 at the University of Michigan

Name: ______________________________

uniqname: ______________________________

UMID: ______________________________

Room: ◯ 1013 DOW ◯ 2246 CSRB ◯ 4721 BBB

Instructions

- This exam consists of 13 problems, worth a total of 130 points, spread across 14 pages (7 sheets of paper). **All problems count towards your Final Exam score; certain problems also count towards your Midterm 1 or Midterm 2 redemption scores.**
- You have 120 minutes to complete this exam, unless you have extended-time accommodations through SSD.
- Write your uniqname in the top right corner of every page in the space provided.
- For free response problems, you must show all of your work (unless otherwise specified), and $\boxed{\text{circle}}$ your final answer. We will not grade work that appears elsewhere, and you may lose points if your work is not shown.
- For multiple choice problems, completely fill in bubbles and square boxes; if we cannot tell which option(s) you selected, you may lose points.
  ◯ A bubble means that you should only select one choice.
  ☐ A square box means you should select all that apply.
- You may refer to **3** two-sided handwritten notes sheet. Other than that, you may not refer to any other resources or technology during the exam (no phones, watches, or calculators).

You are to abide by the University of Michigan/Engineering Honor Code. To receive a grade, please sign below to signify that you have kept the Honor Code pledge.

I have neither given nor received aid on this exam, nor have I concealed any violations of the Honor Code.

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Problem 1 (10 pts) **Counts towards Midterm 1 redemption score**

a) (6 pts) Suppose we'd like to find the optimal constant prediction, w^*, for the constant model $h(x_i) = w$, given a dataset of n values $y_1, y_2, \dots, y_n$. To do so, we minimize mean Bursley error, defined as

$$R_{\text{B}}(w) = \frac{1}{n} \sum_{i=1}^{n} |2y_i - w|^2$$

Suppose the mean of $y_1, y_2, \dots, y_n$ is 20 and the median of $y_1, y_2, \dots, y_n$ is 30.

Which value of w^* minimizes $R_{\text{B}}(w)$ for this dataset? Select one of the answers below, then justify your answer in the box provided.

Hint: Look very closely at the definition of $R_{\text{B}}(w)$. You do not need to re-prove any results from class; you can fully find and explain your answer without using calculus.

(i) Answer: ◯ 10 ◯ 15 ◯ 20 ◯ 30 ◯ 40 ◯ 60

(ii) Justify your answer in the box below.

b) (4 pts) This part does not use any of the numbers from part **a)**.

Recall that the mean absolute error, $R_{\text{abs}}(w)$, of a constant prediction w on a dataset of n values $y_1, y_2, \dots, y_n$ is given by

$$R_{\text{abs}}(w) = \frac{1}{n} \sum_{i=1}^{n} |y_i - w|$$

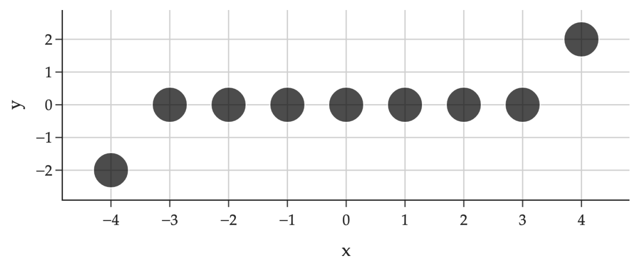
Consider the dataset of 4 values, $1, 3, 5, 9$. Among all integers **not in this dataset**, which **integer** minimizes $R_{\text{abs}}(w)$ for this dataset?

minimizer =

Problem 2 (10 pts) **Counts towards Midterm 1 redemption score**

Let k be a positive integer and let α be a positive real number. Consider the dataset of $n = 2k + 1$ points, $\underbrace{(-k, -\alpha), (-k+1, 0), (-k+2, 0), \ldots, (-1, 0)}_{k \text{ points}}, (0, 0), \underbrace{(1, 0), \ldots, (k-2, 0), (k-1, 0), (k, \alpha)}_{k \text{ points}}$.

Note that the x-values are equally spaced, starting from $-k$ and ending at k. The y-values are all 0, except for the first and last points, which have y-value $-\alpha$ and α, respectively. For example, if $k = 4$ and $\alpha = 2$, the dataset looks like



a) (4 pts) Find $\bar{x}$ and $\bar{y}$, the means of the x- and y-values, respectively. Give your answers as expressions involving k, α, and/or other constants.

$\bar{x} = \boxed{}, \qquad \bar{y} = \boxed{}$

b) (6 pts) Suppose we fit a simple linear regression model to the dataset by minimizing mean squared error. w_1^*, the slope of the regression line, is of the form

$$w_1^* = \frac{A}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

What is the value of A? Select one of the answers below, then justify your answer in the box provided.

(i) Answer: ◯ 0 ◯ α ◯ 2α ◯ $2k\alpha$ ◯ $2k^2\alpha$ ◯ $\frac{2\alpha}{k}$

(ii) Justify your answer in the box below.

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Problem 3 (16 pts) $\boxed{\textbf{Counts towards Midterm 1 redemption score}}$

Consider the vectors $\vec{u} = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ c \end{bmatrix}$, where $c \in \mathbb{R}$ is some constant.

In parts **a)** and **b)**, if there are multiple possible values of c, give just one.

a) (3 pts) Suppose $\vec{u}$ and $\vec{v}$ are orthogonal. Find c. Give your answer as a number with no variables.

$c = \boxed{}$

b) (3 pts) Suppose $\|\vec{v}\| = 4$. Find c. Give your answer as a number with no variables.

$c = \boxed{}$

c) (6 pts) Suppose the projection of $\vec{v}$ onto $\vec{u}$ is $\begin{bmatrix} 1.5 \\ 1.5 \\ 3 \end{bmatrix}$. What is the value of c? Select one of the answers below, then justify your answer in the box provided.

(i) Answer: ○ $1/2$ ○ $3/2$ ○ 2 ○ 4 ○ 6 ○ $6 + \sqrt{41}$ ○ 27

(ii) Justify your answer in the box below.

Recall from the previous page that $\vec{u} = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ c \end{bmatrix}$, where $c \in \mathbb{R}$ is some constant.

d) (4 pts) Suppose $\text{span}(\{\vec{u}, \vec{v}\})$ is the plane $2x + 4y - 3z = 0$. Find c. Show your work, and circle your final answer, which should be a number with no variables. *Hint: While you could compute the cross product, there is no need to — there is a much quicker solution.*

Problem 4 (8 pts) **Counts towards Midterm 2 redemption score**

Let $\vec{u}$ and $\vec{v}$ be as in the previous problem.

a) (4 pts) Suppose that for some value of c, P is the matrix that projects vectors in $\mathbb{R}^3$ onto $\text{span}(\{\vec{u}, \vec{v}\})$. **Select all** true statements below.

☐ $P^2 = P$ ☐ P is invertible ☐ P is orthogonal ☐ P is symmetric

b) (4 pts) Now, suppose $\vec{y} \in \mathbb{R}^3$. Let $\vec{p}$ be the projection of $\vec{y}$ onto $\text{span}(\{\vec{u}, \vec{v}\})$, and let $\vec{e} = \vec{y} - \vec{p}$.

There is no value of c that guarantees that the components of $\vec{e}$ sum to 0, for every $\vec{y} \in \mathbb{R}^3$. That is, it is **not** guaranteed that $e_1 + e_2 + e_3 = 0$ for every $\vec{y} \in \mathbb{R}^3$.

Give a 1-2 sentence English explanation for why it is **not** guaranteed that $e_1 + e_2 + e_3 = 0$ for every $\vec{y} \in \mathbb{R}^3$. *Hint: What* ***would*** *have to be true about $\vec{u}$ and $\vec{v}$ to make this guarantee for every $\vec{y}$?*

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Problem 5 (12 pts) **Counts towards Midterm 2 redemption score**

Consider the $n \times 5$ matrix A, along with a CR decomposition of it, given below.

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 8 & 10 & 12 \\ 5 & 8 & 11 & 14 & 17 \\ 6 & 10 & 14 & 18 & 22 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ n+1 & 2n & 3n-1 & 4n-2 & 5n-3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & ? \\ 3 & ? \\ 4 & ? \\ 5 & ? \\ 6 & ? \\ \vdots & \vdots \\ n+1 & ? \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 1 & \boxed{a} & 0 & c & -1 \\ 0 & \boxed{b} & 1 & d & 2 \end{bmatrix}}_{R}$$

a) (2 pts) Find rank(A). Give your answer as an integer with no variables.

rank$(A) =$ $\boxed{}$

b) (4 pts) Find a and b. Give your answers as numbers with no variables.

$a =$ $\boxed{}$, $b =$ $\boxed{}$

c) (3 pts) State **one** vector in nullsp(A). Give your answer as a vector with no variables. *Hint: It is possible to find a vector in nullsp(A) without using your answer from part **b)**. Try not to rely heavily on your answer from part **b)** in case it's incorrect.*

One vector in nullsp(A) is: $\boxed{}$

d) (3 pts) Fill in the blanks: nullsp(A^T) is a __(i)__-dimensional subspace of __(ii)__.

(i) ○ 2 ○ 3 ○ 4 ○ 5 ○ $n-2$ ○ $n-1$ ○ n

(ii) ○ $\mathbb{R}^2$ ○ $\mathbb{R}^3$ ○ $\mathbb{R}^4$ ○ $\mathbb{R}^5$ ○ $\mathbb{R}^{n-2}$ ○ $\mathbb{R}^{n-1}$ ○ $\mathbb{R}^n$

Problem 6 (4 pts) **Counts towards Midterm 2 redemption score**

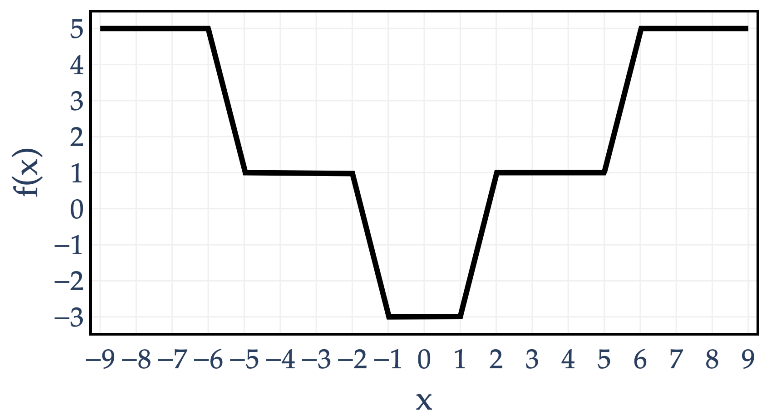
Suppose A and B are both (not necessarily symmetric!) $n \times n$ matrices. Which of the following is $\nabla f(\vec{x})$, the gradient of

$$f(\vec{x}) = (A\vec{x})^T(B\vec{x})$$

○ $2AB\vec{x}$ ○ $A^TB\vec{x}$ ○ $2A^TB\vec{x}$ ○ $2B^TA\vec{x}$ ○ $(A^TB + B^TA)\vec{x}$ ○ $(A^TB - B^TA)\vec{x}$

Problem 7 (6 pts) **Counts towards Midterm 2 redemption score**

Consider the function $f : \mathbb{R} \to \mathbb{R}$ graphed below.



Note that f is a piecewise linear function, with slopes of 0, 4, and -4. The slope changes at the following values of x: $-6, -5, -2, -1, 1, 2, 5, 6$.

Suppose we want to minimize $f(x)$ using gradient descent. There are several values of x such that f is not differentiable at x; if any of our guesses $x^{(0)}, x^{(1)}, x^{(2)}, \ldots$ ever evaluate to one of these values, we say that gradient descent **crashes**.

a) (2 pts) True or False: $f(x)$ is convex on the domain $x \in [-9, 9]$.

○ True ○ False

b) (4 pts) Suppose we choose a learning rate/step size of $\alpha = 0.1$.

Among the options below, which value of $x^{(0)}$ will allow gradient descent to **converge to the global minimum** of $f(x)$ **without crashing**?

If multiple values of $x^{(0)}$ are possible, **select the value that converges the quickest** (i.e. in the fewest number of iterations).

○ 1.4 ○ 1.6 ○ 1.8 ○ 1.9 ○ 2.0

Problem 8 (6 pts) **Counts towards Midterm 2 redemption score**

Suppose we fit a multiple linear regression model **with** an intercept term that predicts the `height` of a wolverine given its `weight` and `color`. The model is fit by minimizing mean squared error.

a) (2 pts) If we one hot encode the color feature **without** dropping any categories, the design matrix X has 6 columns.

How many unique `colors` are there? Give your answer as an integer with no variables.

There are [ ] unique `colors`.

b) (4 pts) Assume that not all wolverines in the dataset have the same `weight`, and that there is at least one wolverine with each color.

What impact would dropping one of the color categories' columns from the design matrix X have? **Select all that apply.**

- ☐ It would decrease the rank of X.
- ☐ It would guarantee that X invertible.
- ☐ It would guarantee that $X^T X$ invertible.
- ☐ It would guarantee the existence of a unique optimal parameter vector $\vec{w}^*$.
- ☐ It would change nullsp(X).
- ☐ It would change colsp(X).

Problem 9 (18 pts)

Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ c & 6 \end{bmatrix}$, where $c \in \mathbb{R}$ is some constant.

Each part asks you to find the values of c, λ_1 (A's **larger eigenvalue**) and λ_2 (A's **smaller eigenvalue**) given the information provided. Your answers should be **numbers with no variables**.

If A only has one unique eigenvalue, put the same number for both λ_1 and λ_2.

Hint: Remember the relationship between the eigenvalues of a matrix and its determinant and trace.

a) (6 pts) A is **not** invertible.

$c = \boxed{}$, $\lambda_1 = \boxed{}$, $\lambda_2 = \boxed{}$

b) (6 pts) A's characteristic polynomial is $p(\lambda) = \lambda^2 - 8\lambda + 7$.

$c = \boxed{}$, $\lambda_1 = \boxed{}$, $\lambda_2 = \boxed{}$

c) (6 pts) A is **not** diagonalizable.

$c = \boxed{}$, $\lambda_1 = \boxed{}$, $\lambda_2 = \boxed{}$

Make sure to place the larger eigenvalue in λ_1 and the smaller eigenvalue in λ_2!

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Problem 10 (12 pts)

Consider the adjacency matrix $A = \begin{bmatrix} 0.4 & 0 & 0.5 \\ 0.4 & 0 & 0.5 \\ a & b & c \end{bmatrix}$ for a Markov chain with three states, where $a, b, c \in \mathbb{R}$ are some constants.

a) (6 pts) Find a, b, and c such that A is a valid adjacency matrix. Give your answers as numbers with no variables.

$a =$ $\boxed{}$, $b =$ $\boxed{}$, $c =$ $\boxed{}$

b) (6 pts) Suppose $\vec{x}^* \in \mathbb{R}^3$ is a vector containing the long-run fraction of time spent in each state. Which of the following vectors is $\vec{x}^*$ and why?

(i) $\vec{x}^*$ is ○ $\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ ○ $\begin{bmatrix} 4/9 \\ 0 \\ 5/9 \end{bmatrix}$ ○ $\begin{bmatrix} 5/16 \\ 5/16 \\ 6/16 \end{bmatrix}$ ○ $\begin{bmatrix} 5/16 \\ 6/16 \\ 5/16 \end{bmatrix}$ ○ $\begin{bmatrix} 3/16 \\ 3/16 \\ 10/16 \end{bmatrix}$

(ii) because $\vec{x}^*$ is the eigenvector of A corresponding to the eigenvalue

○ -1 ○ 0 ○ 0.4 ○ 1 ○ 1.8

Problem 11 (12 pts)

Let A be a 4×4 **symmetric** matrix with eigenvalue decomposition $A = V\Lambda V^{-1}$. Suppose the columns of V are $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$, in that order, and that the columns of V are unit vectors.

a) (2 pts) Suppose $\Lambda = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

True or False: V is guaranteed to be an orthogonal matrix.

◯ True ◯ False

b) (2 pts) Suppose $\Lambda = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

True or False: V is guaranteed to be an orthogonal matrix.

◯ True ◯ False

The rest of this problem does not use any of the information from parts **a)** and **b)**. Suppose k is some positive integer greater than 1, and that

$$\vec{x} = 5\vec{v}_1 - 3\vec{v}_2 - 5\vec{v}_3 + \vec{v}_4$$

and

$$A^k\vec{x} = 40\vec{v}_1 - 81\vec{v}_2 + 64\vec{v}_4$$

c) (6 pts) What is the value of k? Select one of the answers below, then justify your answer in the box provided. *Hint: If $A = V\Lambda V^{-1}$, what is A^k?*

(i) Answer: ◯ 2 ◯ 3 ◯ 4 ◯ 5

(ii) Justify your answer in the box below.

d) (2 pts) Fill in the blank: as $k \to \infty$, the direction of $A^k\vec{x}$ approaches the direction of...

◯ $\vec{v}_1$ ◯ $\vec{v}_2$ ◯ $\vec{v}_3$ ◯ $\vec{v}_4$

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Problem 12 (12 pts)

Suppose $\tilde{X}$ is a 24×3 matrix whose columns are mean-centered (i.e. have a mean of 0). Let $\tilde{X} = U\Sigma V^T$ be the singular value decomposition of $\tilde{X}$, where

$$\tilde{X} = U \underbrace{\begin{bmatrix} 12 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} & 0 \\ \cdots & \vec{v}_2^T & \cdots \\ 0 & 0 & 1 \end{bmatrix}}_{V^T}$$

a) (2 pts) Find $\text{rank}(\tilde{X})$. Give your answer as an integer with no variables.

$\text{rank}(\tilde{X}) =$ $\boxed{}$

b) (3 pts) It is possible to find $\vec{v}_2^T$, the second row of V^T, solely using the information provided (without knowing any of the values in $\tilde{X}$). In one English sentence, **explain how** to find it.

c) (2 pts) True or False: There exists some vector $\vec{z} \in \mathbb{R}^{24}$ such that $\tilde{X}\tilde{X}^T\vec{z} = 2\vec{z}$.

◯ True ◯ False ◯ Impossible to tell

d) (5 pts) What is the largest possible variance of the components of $\tilde{X}\vec{w}$, where $\vec{w} \in \mathbb{R}^3$ is a unit vector? Select one of the answers below, then justify your answer in the box provided.

(i) Answer: ◯ 1 ◯ 2 ◯ 6 ◯ 12 ◯ 24 ◯ 144

(ii) Justify your answer in the box below.

Problem 13 (4 pts)

What is one topic you studied a lot for that wasn't on the Final Exam? **Blank answers will receive no credit!**

Congrats on completing the Final Exam for EECS 245! We'll really miss you; please stay in touch.

Feel free to draw us a picture about EECS 245 in the box below.