Midterm 1

EECS 245, Fall 2025 at the University of Michigan

Name: ______________________________

uniqname: ______________________________

UMID: ______________________________

Room: ◯ 1013 DOW ◯ 2725 BBB

Instructions

- This exam consists of 8 problems, worth a total of 100 points, spread across 12 pages (6 sheets of paper).
- You have 80 minutes to complete this exam, unless you have extended-time accommodations through SSD.
- Write your uniqname in the top right corner of each page.
- For free response problems, you must show all of your work (unless otherwise specified), and $\boxed{\text{circle}}$ your final answer. We will not grade work that appears elsewhere, and you may lose points if your work is not shown.
- For multiple choice problems, completely fill in bubbles and square boxes; if we cannot tell which option(s) you selected, you may lose points.
  ◯ A bubble means that you should only select one choice.
  ☐ A square box means you should select all that apply.
- You may refer to a single two-sided handwritten notes sheet. Other than that, you may not refer to any other resources or technology during the exam (no phones, watches, or calculators).

You are to abide by the University of Michigan/Engineering Honor Code. To receive a grade, please sign below to signify that you have kept the Honor Code pledge.

I have neither given nor received aid on this exam, nor have I concealed any violations of the Honor Code.

This page has been intentionally left blank.

Problem 1: Consider the Following... (15 pts)

Consider the following dataset of $n = 9$ values.

y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
7	8	10	10	11	13	14	17	27

Suppose we'd like to find the optimal parameter, w^*, for the constant model $h(x_i) = w$, given this dataset of 9 values.

In parts **a)** through **f)**, choose the empirical risk function $R(w)$ that the given value of w^* is the minimizer of, for this particular dataset. If you believe the given value of w^* does not minimize any of the five options, select N/A.

- **Option 1**: $R(w) = \frac{1}{n}\sum_{i=1}^{n}(y_i - w)^2$
- **Option 2**: $R(w) = \frac{1}{n}\sum_{i=1}^{n}(27y_i - 13w)^2$
- **Option 3**: $R(w) = \frac{1}{n}\sum_{i=1}^{n}13|y_i - w|$
- **Option 4**: $R(w) = \frac{1}{n}\sum_{i=1}^{n}\begin{cases}13 & \text{if } y_i = w \\ 27 & \text{if } y_i \neq w\end{cases}$
- **Option 5**: $R(w) = \lim_{p\to\infty}\frac{1}{n}\sum_{i=1}^{n}|y_i - w|^p$

a) (2.5 pts) **10** is the value of w that minimizes...
◯ Option 1 ◯ Option 2 ◯ Option 3 ◯ Option 4 ◯ Option 5 ◯ N/A

b) (2.5 pts) **11** is the value of w that minimizes...
◯ Option 1 ◯ Option 2 ◯ Option 3 ◯ Option 4 ◯ Option 5 ◯ N/A

c) (2.5 pts) **12** is the value of w that minimizes...
◯ Option 1 ◯ Option 2 ◯ Option 3 ◯ Option 4 ◯ Option 5 ◯ N/A

d) (2.5 pts) **13** is the value of w that minimizes...
◯ Option 1 ◯ Option 2 ◯ Option 3 ◯ Option 4 ◯ Option 5 ◯ N/A

e) (2.5 pts) **17** is the value of w that minimizes...
◯ Option 1 ◯ Option 2 ◯ Option 3 ◯ Option 4 ◯ Option 5 ◯ N/A

f) (2.5 pts) **27** is the value of w that minimizes...
◯ Option 1 ◯ Option 2 ◯ Option 3 ◯ Option 4 ◯ Option 5 ◯ N/A

Problem 2: Absolute Madness (17 pts)

Consider a dataset of $n = 8$ values, where

$$y_1 = 1, \quad y_2 = y_3 = 4, \quad y_4 = y_5 = y_6 = \alpha, \quad y_7 = y_8 = 20$$

and $4 < \alpha < 20$.

As usual, let $R_{\text{abs}}(w)$ represent the mean absolute error of a constant prediction w on this dataset of 8 values.

a) (3 pts) Is the value of w^*, the minimizer of $R_{\text{abs}}(w)$, unique? Select and fill out one option below.

◯ The value of w^* is unique, and is equal to $\boxed{}$.

◯ The value of w^* is not unique; any value between $\boxed{}$ and $\boxed{}$ is a minimizer.

b) (6 pts) Find the value of $R_{\text{abs}}(\alpha)$, for any valid choice of α. Show your work, and $\boxed{\text{circle}}$ your final answer, which should be an expression involving α and other constants, but no other variables, and no summation notation.

Recall,

$$y_1 = 1, \quad y_2 = y_3 = 4, \quad y_4 = y_5 = y_6 = \alpha, \quad y_7 = y_8 = 20$$

where $4 < \alpha < 20$.

c) (8 pts) Let the minimum possible value of $R_{\text{abs}}(w)$ be M. Given that

$$R_{\text{abs}}(20) - M = \frac{9}{2}$$

find the value of α. Show your work, and circle your final answer, which should be a number with no variables.

Hint: It's possible to answer this without using your answer from the previous part.

Problem 3: Spreading Your Wings (12 pts)

Consider a dataset of n points, $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, where

- the means of $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are 15 and 5, respectively
- the variances of $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are σ_x^2 and σ_y^2, respectively
- the correlation coefficient between $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ is r

We define a new set of values, $z_1, z_2, \ldots, z_n$, as follows:

$$z_i = 3x_i - y_i, \quad i = 1, 2, \ldots, n$$

a) (4 pts) Suppose we fit a simple linear regression line to the dataset $(x_1, z_1), (x_2, z_2), \ldots, (x_n, z_n)$ by minimizing mean squared error. Note that z is the variable being predicted, not y. Let $h(x_i)$ represent the corresponding line.

What is the value of $h(15)$? Your answer should be a number with no variables.

$h(15) =$ $\boxed{}$

b) (8 pts) σ_z^2, the variance of $z_1, z_2, \ldots, z_n$, can be written in the form $\sigma_z^2 = 9\sigma_x^2 + \sigma_y^2 + C$.

(i) What is the value of C?

○ $-6\sigma_x\sigma_y$ ○ $6\sigma_x\sigma_y$ ○ $-6r\sigma_x\sigma_y$ ○ $6r\sigma_x\sigma_y$ ○ $-6nr\sigma_x\sigma_y$ ○ $6nr\sigma_x\sigma_y$

(ii) Show your work in the box below. English explanations are not enough.

Problem 4: Mission Impossible (12 pts)

a) (6 pts) Suppose $\vec{u}, \vec{v} \in \mathbb{R}^n$ are **non-zero** vectors, and suppose that

$$|\vec{u} \cdot \vec{v}| = \|\vec{u}\| \|\vec{v}\|$$

For each statement below, determine whether it is impossible, possible, or guaranteed to be true, given the above assumptions. **Select exactly one option from each row**. The first statement has been done for you as an example.

	statement	**impossible?**	**possible?**	**guaranteed?**
(i)	$\|\vec{u}\| = 5$	○	●	○
(ii)	$\vec{u}$ and $\vec{v}$ are orthogonal	○	○	○
(iii)	$\|\vec{u} - \vec{v}\| = 0$	○	○	○
(iv)	$\vec{u}$ and $\vec{v}$ span a 1-dimensional subspace of $\mathbb{R}^n$	○	○	○
(v)	$\vec{u}$ and $\vec{v}$ span a 2-dimensional subspace of $\mathbb{R}^n$	○	○	○
(vi)	$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$	○	○	○

b) (6 pts) Suppose $\vec{w}, \vec{z} \in \mathbb{R}^n$. Given that $\|\vec{w}\| = \|\vec{z}\| = \|\vec{w} - \vec{z}\| = 1$, find $\|\vec{w} + \vec{z}\|$. Show your work, and $\boxed{\text{circle}}$ your final answer, which should be a number with no variables.

Problem 5: Back to Normal (12 pts)

Consider the orthogonal vectors $\vec{u}_1 = \begin{bmatrix} 13 \\ -3 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}$, and $\vec{u}_3 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$.

a) (4 pts) Find the equation of the plane spanned by $\vec{u}_2$ and $\vec{u}_3$ in standard form, i.e. $ax + by + cz + d = 0$. Circle your final answer.

b) (8 pts) There is one value of k such that the projection of $\vec{x} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$ onto $\vec{u}_k$ is just $\vec{u}_k$ itself.

(i) What is the value of k? ◯ 1 ◯ 2 ◯ 3

(ii) Show your work in the box below. English explanations are not enough.

Problem 6: Needed Me (11 pts)

Suppose $\vec{x} = \begin{bmatrix} c \\ 1 \\ 0 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 1 \\ c \\ 1 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} 0 \\ 1 \\ c \end{bmatrix}$, where $c \in \mathbb{R}$ is a constant.

a) (8 pts) Find a **positive value** of c such that $\vec{x}$, $\vec{y}$, and $\vec{z}$ are linearly **dependent**. Show your work, and circle your final answer, which should be a positive number with no variables.

b) (3 pts) Provide one **other** value of c (that is, not your answer from the previous part) such that $\vec{x}$, $\vec{y}$, and $\vec{z}$ are linearly **dependent**. Your answer should be a number with no variables.

other value of $c =$

Problem 7: High Definition (12 pts)

Suppose $\vec{x}_1, \vec{x}_2, \dots \vec{x}_{12}$ are 12 non-zero vectors in $\mathbb{R}^7$. Furthermore, suppose:

- $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ span a 2-dimensional subspace of $\mathbb{R}^7$.
- $\vec{x}_4$, $\vec{x}_5$, and $\vec{x}_6$ span **the same** 2-dimensional subspace of $\mathbb{R}^7$ as $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$, i.e.

$$\text{span}(\{\vec{x}_4, \vec{x}_5, \vec{x}_6\}) = \text{span}(\{\vec{x}_1, \vec{x}_2, \vec{x}_3\})$$

a) (4 pts) Let r be the dimension of the subspace of $\mathbb{R}^7$ spanned by $\vec{x}_1, \vec{x}_2, \dots \vec{x}_{12}$. What are the smallest and largest possible values of r? Your answers should be integers with no variables.

smallest possible value of $r =$ ☐ largest possible value of $r =$ ☐

b) (4 pts) Which of the following **could** form a basis for $\mathbb{R}^7$? Select all that apply. Blank answers will receive no credit.

- ☐ $\{\vec{x}_7, \vec{x}_8, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$
- ☐ $\{\vec{x}_6, \vec{x}_7, \vec{x}_8, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$
- ☐ $\{\vec{x}_1, \vec{x}_5, \vec{x}_8, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$
- ☐ $\{\vec{x}_1, \vec{x}_2, \vec{x}_5, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$
- ☐ $\{\vec{x}_1, \vec{x}_2, \vec{x}_8, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$

c) (4 pts) Suppose the intersection of $\text{span}(\{\vec{x}_1, \vec{x}_2\})$ and $\text{span}(\{\vec{x}_4, \vec{x}_5\})$ is a line (i.e. a 1-dimensional subspace) in $\mathbb{R}^7$. Which of the following **must** be true? Select all that apply. Blank answers will receive no credit.
Hint: Don't forget the assumptions introduced at the start of the problem.

- ☐ $\vec{x}_2$, $\vec{x}_4$, and $\vec{x}_5$ can all be written as scalar multiples of $\vec{x}_1$.
- ☐ The set $\{\vec{x}_2, \vec{x}_4\}$ is linearly independent.
- ☐ The set $\{\vec{x}_3, \vec{x}_4\}$ is linearly independent.
- ☐ The set $\{\vec{x}_3, \vec{x}_6\}$ is linearly independent.
- ☐ None of the above.

Problem 8: Worst-Case Scenario (8 pts)

Suppose a, b, c, d, e are positive real numbers. Find the **largest** real number T such that it's guaranteed that

$$(a+b+c+d+e)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right) \geq T$$

Think of T as the "best possible lower bound". For instance, we know that the expression on the left-hand side above must be greater than or equal to 0, since a, b, c, d, e are all positive, but $T = 0$ is not the answer, since there's a larger value of T that also guarantees the inequality holds.

Show your work, and $\boxed{\text{circle}}$ your final answer, which should be a number with no variables.

Hint: Use the Cauchy-Schwarz inequality.

(1 pt) Congrats on finishing Midterm 1! Here's a free point.

Feel free to draw us a picture about EECS 245 in the box below.