# Midterm 2

EECS 245, Fall 2025 at the University of Michigan

**Name:** ______________________________

**uniqname:** ______________________________

**UMID:** ______________________________

**Room:** ◯ 1365 LCSIB ◯ 2901 BBB

**Instructions**

- This exam consists of 7 problems, worth a total of 100 points, spread across 12 pages (6 sheets of paper).
- You have 80 minutes to complete this exam, unless you have extended-time accommodations through SSD.
- Write your uniqname in the top right corner of every page in the space provided.
- For free response problems, you must show all of your work (unless otherwise specified), and circle your final answer. We will not grade work that appears elsewhere, and you may lose points if your work is not shown.
- For multiple choice problems, completely fill in bubbles and square boxes; if we cannot tell which option(s) you selected, you may lose points.
  ◯ A bubble means that you should only select one choice.
  ☐ A square box means you should select all that apply.
- You may refer to a single two-sided handwritten notes sheet. Other than that, you may not refer to any other resources or technology during the exam (no phones, watches, or calculators).

You are to abide by the University of Michigan/Engineering Honor Code. To receive a grade, please sign below to signify that you have kept the Honor Code pledge.

*I have neither given nor received aid on this exam, nor have I concealed any violations of the Honor Code.*

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## Problem 1: Getting Started (12 pts)

**a)** (3 pts) Let $A = \begin{bmatrix} 2 & 4 \\ -3 & -7 \end{bmatrix}$. Find $\det(A)$, the determinant of $A$. Give your answer as an integer.

$\det(A) = \boxed{\phantom{xxxx}}$

**b)** (3 pts) Using $A$ from part **a)**, find $A^{-1}$, the inverse of $A$. Fully simplify your answer, i.e. don't leave any constants out front.

$A^{-1} = \boxed{\phantom{xxxx}}$

**c)** (2 pts) Let $B = \begin{bmatrix} -1 & 2 & -1 \\ 3 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. What is the **first column** of $B^{-1}$, the inverse of $B$?

○ $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} -1 \\ 1/2 \\ -1 \end{bmatrix}$ ○ $\begin{bmatrix} -1 \\ 1/3 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} -1/3 \\ 1/3 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1/3 \\ -1/3 \\ 0 \end{bmatrix}$ ○ $B$ is not invertible

**d)** (4 pts) This part is independent of the previous parts (i.e. don't use the specific $A$ or $B$ from above).

**Select all** true statements below.

- ☐ If $A$ and $B$ are both matrices such that $AB = I$, then $A$ and $B$ are both invertible.
- ☐ If $A$ and $B$ are both invertible matrices, then $(A^TB)^{-1} = \left((B^{-1})^TA^{-1}\right)^T$.
- ☐ If $A$ is an invertible matrix, then $\text{rank}(A) = \text{rank}(A^{-1})$.
- ☐ If $A$, $B$, and $C$ are all symmetric matrices, then $AB + C$ is also symmetric.

**Problem 2: Space Jam (20 pts)**

Let $X = \begin{bmatrix} 1 & -4 & 2 & 2 & 0 \\ 0 & 0 & -3 & 3 & 0 \\ 1 & -4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

**a)** (4.5 pts) Determine the values of each of the following. Give your answers as integers.

$\dim(\text{colsp}(X)) =$ $\boxed{\phantom{00}}$ $\dim(\text{nullsp}(X)) =$ $\boxed{\phantom{00}}$

$\dim(\text{colsp}(X^T)) =$ $\boxed{\phantom{00}}$ $\dim(\text{nullsp}(X^T)) =$ $\boxed{\phantom{00}}$

**b)** (3.5 pts) Suppose $\vec{y} \in \mathbb{R}^4$. How many solutions $\vec{v} \in \mathbb{R}^5$ are there to the system of equations $X\vec{v} = \vec{y}$? **Select all** possibilities, since the answer may depend on $\vec{y}$.

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ Infinitely many

**c)** (6 pts) For some $\vec{y} \in \mathbb{R}^4$, the vector $\vec{w}' = \begin{bmatrix} 8 \\ 0 \\ 0 \\ 3 \\ 11 \end{bmatrix}$ is such that $X\vec{w}'$ is the vector in $\text{colsp}(X)$ that is closest to $\vec{y}$. State **one other** vector $\vec{\beta}$ such that $X\vec{\beta} = X\vec{w}'$. Show your work, and $\boxed{\text{circle}}$ your final answer, which should be a vector with five entries and no variables.

Recall, $X = \begin{bmatrix} 1 & -4 & 2 & 2 & 0 \\ 0 & 0 & -3 & 3 & 0 \\ 1 & -4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

**d)** (6 pts) Find a basis for $\text{nullsp}(X^T)$ (**not** $\text{nullsp}(X)$). Show your work, and $\boxed{\text{circle}}$ your final answer, which should be a list of vectors.

**Problem 3: Nilpotence (12 pts)**

Suppose $A$ is an $n \times n$ matrix such that $A^2 = 0_{n\times n}$, where $0_{n\times n}$ is an $n \times n$ matrix of all zeros.

**a)** (6 pts) Prove that if $\vec{x} \in \text{colsp}(A)$, then $\vec{x} \in \text{nullsp}(A)$.

**b)** (6 pts) In part **a)**, you showed that colsp$(A)$ is a subset of nullsp$(A)$. Using this fact, find the **maximum** possible value of rank$(A)$. Show your work and $\boxed{\text{circle}}$ your final answer, which should be an expression involving $n$ and/or constants.

## Problem 4: Poly Wants a Cracker (18 pts)

Suppose we'd like to fit the model $\boxed{h(x_i) = w_0 + w_1 x_i + w_2 x_i^2}$ by minimizing mean squared error. We use an observation vector $\vec{y} \in \mathbb{R}^n$, but instead of using the regular design matrix $X$,

$$X = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{x}^{(0)} & \vec{x}^{(1)} & \vec{x}^{(2)} \\ | & | & | \end{bmatrix}$$

we use the **centered** design matrix $Z$ (where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the mean of the $x$'s).

$$Z = \begin{bmatrix} 1 & x_1 - \bar{x} & (x_1 - \bar{x})^2 \\ 1 & x_2 - \bar{x} & (x_2 - \bar{x})^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n - \bar{x} & (x_n - \bar{x})^2 \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{z}^{(0)} & \vec{z}^{(1)} & \vec{z}^{(2)} \\ | & | & | \end{bmatrix}$$

**a)** (6 pts) It turns out that $\text{colsp}(Z) = \text{colsp}(X)$. To show this, fill in the blanks below to express $\vec{z}^{(2)}$ (the third column of $Z$) as a linear combination of $X$'s columns. Each box should be filled with an expression involving $\bar{x}$, $n$, and/or constants.

$$\vec{z}^{(2)} = \boxed{\phantom{xxxxxx}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \boxed{\phantom{xxxxxx}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \boxed{\phantom{xxxxxx}} \begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_n^2 \end{bmatrix}$$

**b)** (6 pts) In this part only, assume that the values $x_1, x_2, ..., x_n$ are each either 1 or 0. For some specific values $x_1, x_2, ..., x_n$, the matrix $P$ that projects vectors in $\mathbb{R}^n$ onto $\text{colsp}(Z)$ is given by

$$P = \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

(i) What is the rank of $Z$? Give your answer as an integer. $\text{rank}(Z) = \boxed{\phantom{xxxxxx}}$

(ii) Which specific values of $x_1, x_2, ..., x_n$ result in $P$ being the matrix above? Give your answer as a list of values, in the order $x_1$, then $x_2$, then $x_3$, etc. (If there are multiple possible answers, just give one.)

$\boxed{\phantom{xxxxxxxxxxxxxxxxxxxxxxxx}}$

Recall, $Z = \begin{bmatrix} 1 & x_1 - \bar{x} & (x_1 - \bar{x})^2 \\ 1 & x_2 - \bar{x} & (x_2 - \bar{x})^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n - \bar{x} & (x_n - \bar{x})^2 \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{z}^{(0)} & \vec{z}^{(1)} & \vec{z}^{(2)} \\ | & | & | \end{bmatrix}$.

**c)** (6 pts) Let $\vec{\beta}^* = \begin{bmatrix} \beta_0^* \\ \beta_1^* \\ \beta_2^* \end{bmatrix}$ be a solution to the normal equations for $Z$ and $\vec{y}$. Show that

$$\beta_0^* = \bar{y} - \beta_2^* \sigma_x^2$$

where $\sigma_x^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$ is the variance of the $x$'s, and $\bar{y}$ is the mean of the $y$'s. *Hint: Use the fact that* $\sum_{i=1}^{n} (x_i - \bar{x}) = 0$. *What is the error vector? Is it orthogonal to something useful?*

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## Problem 5: Ortho...dontist? (12 pts)

Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$.

**a)** (6 pts) Find a matrix $Q$ such that $\text{colsp}(Q) = \text{colsp}(A)$ and $Q^T Q = I$. Show your work and circle your final answer, which should be a matrix with two columns and no variables. *Hint: One of the columns may involve square roots.*

**b)** (2 pts) True or False: The matrix $Q$ you found above is an orthogonal matrix.

◯ True ◯ False

**c)** (4 pts) Let $R = \begin{bmatrix} r_1 & \boxed{r_2} \\ \boxed{r_3} & r_4 \end{bmatrix}$ be a $2 \times 2$ matrix such that $A = QR$, where $Q$ is the matrix you found above.

Find $r_2$ and $r_3$. Give your answers as scalars without variables.

$r_2 =$ ____________, $r_3 =$ ____________

## Problem 6: Quadratus Formulus (14 pts)

Let $f(\vec{x}) = \frac{1}{2}\vec{x}^T S \vec{x} - \vec{b}^T \vec{x}$, where $S$ is a symmetric $n \times n$ matrix and $\vec{b} \in \mathbb{R}^n$.

**a)** (4 pts) Find $\nabla f(\vec{x})$, the gradient of $f(\vec{x})$. Show your work, and circle your final answer, which should be an expression in terms of $\vec{x}$, $S$, $\vec{b}$, and/or constants. *Hint: There's no need to re-prove gradient rules from class.*

**b)** (2 pts) True or False: As long as $S$ is invertible, if $\nabla f(\vec{a}) = \vec{0}$, then $\vec{a}$ is a global minimum of $f(\vec{x})$.

○ True ○ False

**c)** (2 pts) True or False: As long as all of the components of $S$ are positive real numbers, if $\nabla f(\vec{a}) = \vec{0}$, then $\vec{a}$ is a global minimum of $f(\vec{x})$.

○ True ○ False

**d)** (6 pts) We'd like to use gradient descent to minimize $f(\vec{x})$. Suppose $S = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, and we use a learning rate of $\alpha = 1$. After one iteration of gradient descent, we have $\vec{x}^{(1)} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$. What was our initial guess, $\vec{x}^{(0)}$? Show your work, and circle your final answer, which should be a vector with two entries and no variables.

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## Problem 7: Complexity (10 pts)

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a convex function.

**a)** (4 pts) Find scalars $a$ and $b$ such that $f(3) \leq af(2) + bf(6)$. Show your work and $\boxed{\text{circle}}$ your final answer, which should be a pair of scalars.

**b)** (6 pts) Using the result from part **a)**, prove that $f(3) + f(5) \leq f(2) + f(6)$.

(2 pts) Congrats on finishing Midterm 2! Here are two free points.

Feel free to draw us a picture about EECS 245 in the box below.