Solutions from review session
(some extra details added after)

Mock Midterm 1

EECS 245, Fall 2025 at the University of Michigan

Name: ______________________________

uniqname: ______________________________

UMID: ______________________________

Room: ◯ 1365 LCSIB ◯ 2901 BBB

Instructions

- This exam consists of 7 questions. **On the real midterm, we will also state the total number of points for each question. We make no guarantees on the number of questions, points, or specific questions on the real midterm.**
- You have 80 minutes to complete this exam, unless you have extended-time accommodations through SSD.
- Write your uniqname in the top right corner of each page in the space provided.
- For free response problems, you must show all of your work (unless otherwise specified), and circle your final answer. We will not grade work that appears elsewhere, and you may lose points if your work is not shown.
- For multiple choice problems, completely fill in bubbles and square boxes; if we cannot tell which option(s) you selected, you may lose points.
  ◯ A bubble means that you should only select one choice.
  ☐ A square box means you should select all that apply.
- You may refer to a single two-sided handwritten notes sheet. Other than that, you may not refer to any other resources or technology during the exam (no phones, watches, or calculators).

You are to abide by the University of Michigan/Engineering Honor Code. To receive a grade, please sign below to signify that you have kept the Honor Code pledge.

I have neither given nor received aid on this exam, nor have I concealed any violations of the Honor Code.

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Review Ch. 1.3,
3 step modeling recipe

Problem 1: Doubling Down

Suppose we'd like to find the optimal parameter, w^*, for the constant model $h(x_i) = w$, given a dataset of n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. To do so, we use the **doubly squared** loss function, L_{ds}, defined below.

$$L_{\text{ds}}(y_i, w) = (y_i^2 - w^2)^2$$

a) Find $\frac{dR_{\text{ds}}}{dw}$, the derivative of average doubly squared loss (i.e. the empirical risk) with respect to w.

$$R_{ds}(w) = \frac{1}{n}\sum_{i=1}^{n}(y_i^2 - w^2)^2$$

$$\frac{dR_{ds}}{dw} = \frac{1}{n}\sum_{i=1}^{n}\Big(\underbrace{\frac{d}{dw}(y_i^2 - w^2)^2}_{\text{chain rule}}\Big)$$

$$= \frac{1}{n}\sum_{i=1}^{n} 2(y_i^2 - w^2)(-2w)$$

independent of sum

$$= -\frac{4w}{n}\sum_{i=1}^{n}(y_i^2 - w^2)$$

b) Show that the value of w that minimizes average doubly squared loss is

$$w^* = \sqrt{\frac{1}{n}\sum_{i=1}^{n} y_i^2}$$

"quadratic mean", QM

$$\frac{dR}{dw} = 0 \rightarrow -\frac{4w}{n}\sum_{i=1}^{n}(y_i^2 - w^2) = 0$$

$$\sum_{i=1}^{n}(y_i^2 - w^2) = 0$$

$$\sum_{i=1}^{n} y_i^2 - \underbrace{\sum_{i=1}^{n} w^2}_{nw^2} = 0$$

$$\sum_{i=1}^{n} y_i^2 = nw^2$$

$$\rightarrow w^* = \sqrt{\frac{1}{n}\sum_{i=1}^{n} y_i^2}$$

Followup could have been:

Prove that for any dataset,

$$QM \geq AM, \text{ i.e.}$$

$$\underbrace{\sqrt{\frac{1}{n}\sum_{i=1}^{n} y_i^2}}_{\text{quadratic mean}} \geq \underbrace{\frac{1}{n}\sum_{i=1}^{n} y_i}_{\text{arithmetic (regular) mean}}$$

(last part)

Hint: use the fact that the variance, $\sigma_y^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 \geq 0$

can show $\frac{1}{n}\sum(y_i-\bar{y})^2 = \underbrace{\frac{1}{n}\sum_{i=1}^{n} y_i^2}_{QM^2} - \underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)^2}_{AM^2}$

since variance ≥ 0, and

variance $= QM^2 - AM^2$,

we have

$$QM^2 - AM^2 \geq 0$$

$$QM^2 \geq AM^2$$

safe since $QM \geq 0$

$$QM \geq AM$$

Review Ch. 1.3

$y_1, y_2, y_3, z_1, z_2, z_3, z_4, z_5$

Problem 2: Absolutely...

Consider a dataset of 3 values, $y_1 < y_2 < y_3$, with a mean of 2. Let

$$Y_{\text{abs}}(w) = \frac{1}{3}\sum_{i=1}^{3} |y_i - w|$$

represent the mean absolute error of a constant prediction w on this dataset of 3 values.

Similarly, consider another dataset of 5 values, $z_1 < z_2 < z_3 < z_4 < z_5$, with a mean of 12. Let

$$Z_{\text{abs}}(w) = \frac{1}{5}\sum_{i=1}^{5} |z_i - w|$$

represent the mean absolute error of a constant prediction w on this dataset of 5 values.

Suppose that $y_3 < z_1$, and that $T_{\text{abs}}(w)$ represents the mean absolute error of a constant prediction w on the combined dataset of 8 values, $y_1, y_2, y_3, z_1, z_2, z_3, z_4, z_5$.

a) Fill in the blanks to complete the sentence: Median minimizes MAE

___(i)___ minimizes $Y_{\text{abs}}(w)$, ___(ii)___ minimizes $Z_{\text{abs}}(w)$, and ___(iii)___ minimizes $T_{\text{abs}}(w)$.

Note that in the options below, $[a, b]$ represents the range of values between a and b, including both a and b.

(i) ○ y_1 ○ any value in $[y_1, y_2]$ ● y_2 ○ y_3 ○ z_1

(ii) ○ z_1 ○ z_2 ○ any value in $[z_2, z_3]$ ○ any value in $[z_3, z_4]$ ● z_3

(iii) ○ y_2 ○ y_3 ○ any value in $[y_3, z_1]$ ● any value in $[z_1, z_2]$ ○ any value in $[z_2, z_3]$

since # points is even

b) For any value w, it's true that

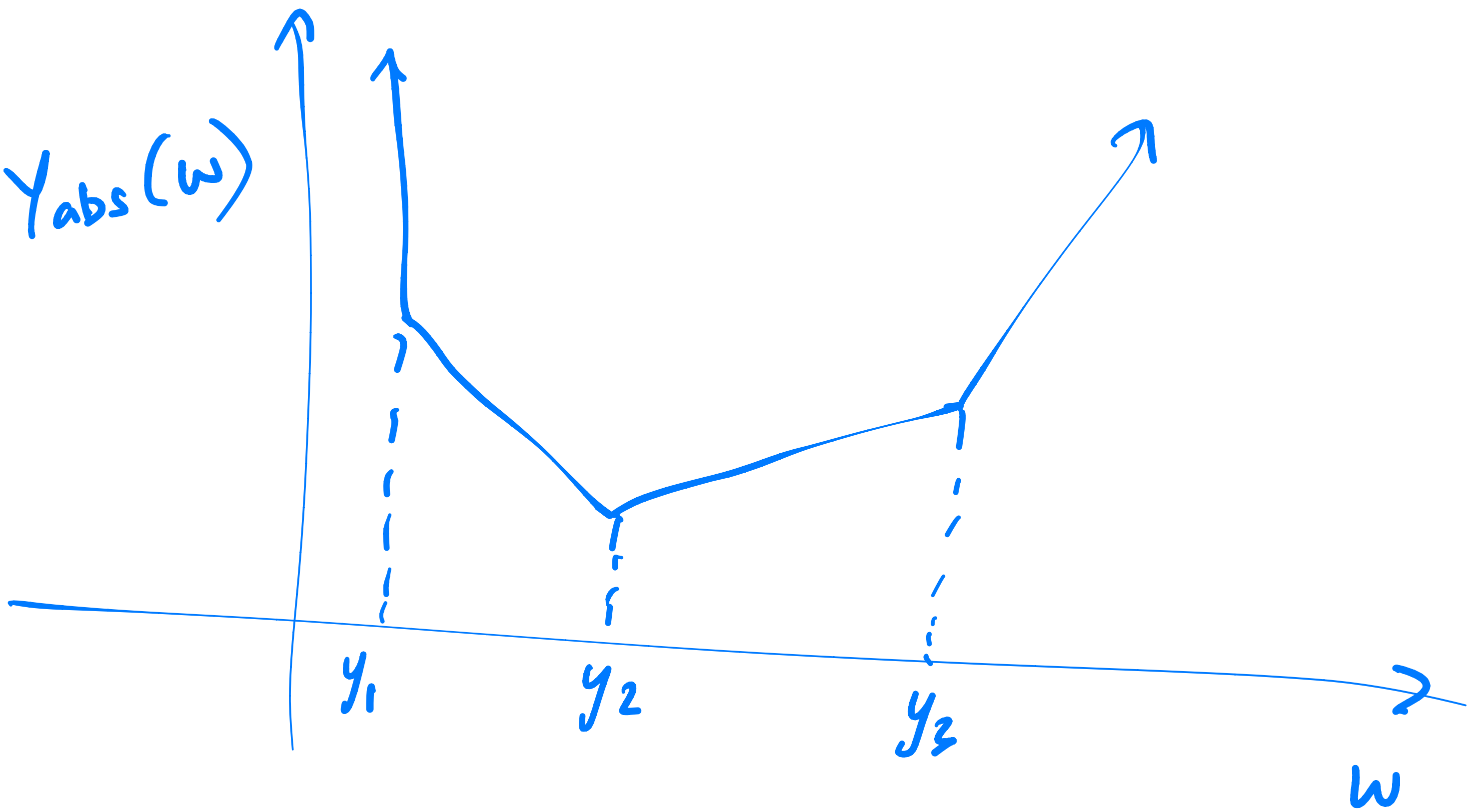
$$T_{\text{abs}}(w) = \alpha Y_{\text{abs}}(w) + \beta Z_{\text{abs}}(w)$$

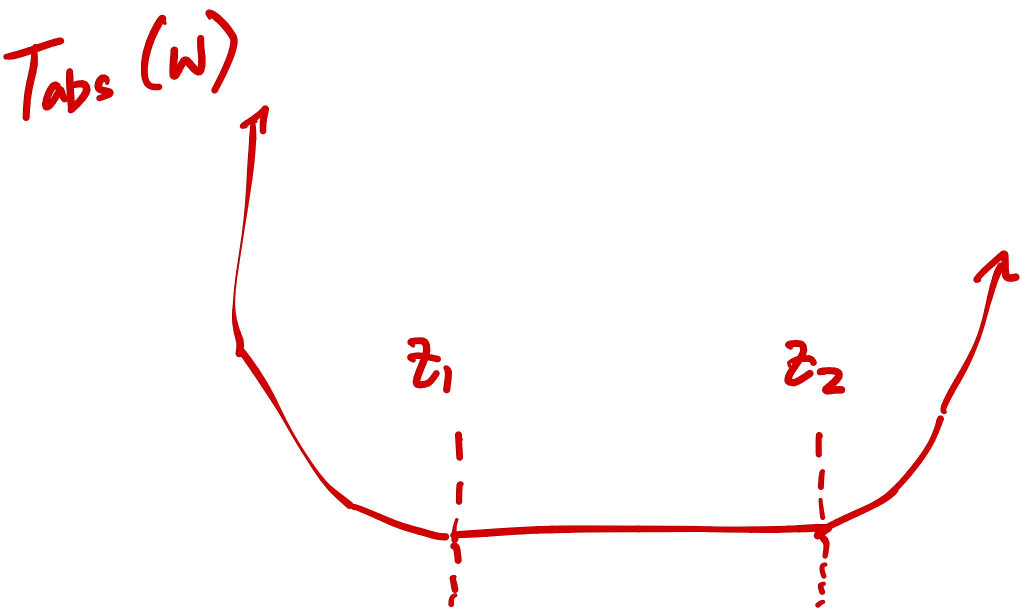
for some constants α and β. Determine the values of α and β. Both answers should be integers or simplified fractions with no variables.

$\alpha = \boxed{3/8}$ $\beta = \boxed{5/8}$

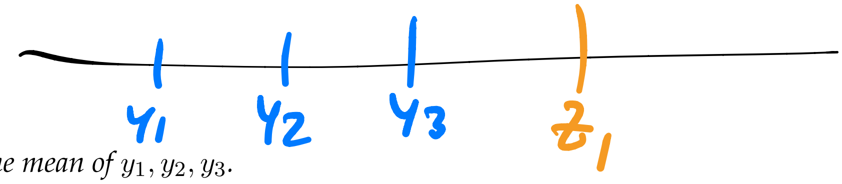
$$T_{abs}(w) = \frac{|y_1 - w| + |y_2 - w| + |y_3 - w| + |z_1 - w| + |z_2 - w| + \dots + |z_5 - w|}{8}$$

$Y_{abs}(w)$





$$T_{abs}(w) = \underbrace{\underbrace{|y_1 - w| + |y_2 - w| + |y_3 - w|}_{3\,Y_{abs}(w)} + \underbrace{|z_1 - w| + |z_2 - w| + \dots + |z_5 - w|}_{5\,Z_{abs}(w)}}_{8}$$



c) Show that $Y_{\text{abs}}(z_1) = z_1 - 2$.
Hint: Use the fact that you know the mean of y_1, y_2, y_3.

$$Y_{abs}(z_1) = \frac{|y_1 - z_1| + |y_2 - z_1| + |y_3 - z_1|}{3}$$

$$= \frac{z_1 - y_1 + z_1 - y_2 + z_1 - y_3}{3}$$

$$= \frac{3z_1}{3} - \frac{y_1 + y_2 + y_3}{3} \qquad \bar{Y}$$

$$= z_1 - 2$$

d) Suppose the minimum possible **output** of $T_{\text{abs}}(w)$ in the full dataset of 8 values is 6. What is the value of z_1?
Hint: You'll need to use the answers to the previous parts.

○ -2 ○ 0 ○ 2 ● 3 ○ 5 ○ 6 ○ 7 ○ 9

$$\underbrace{T_{abs}(z_1) = \frac{3}{8}\underbrace{Y_{abs}(z_1)}_{\text{part c}} + \frac{5}{8} Z_{abs}(z_1)}_{\text{part b}}$$

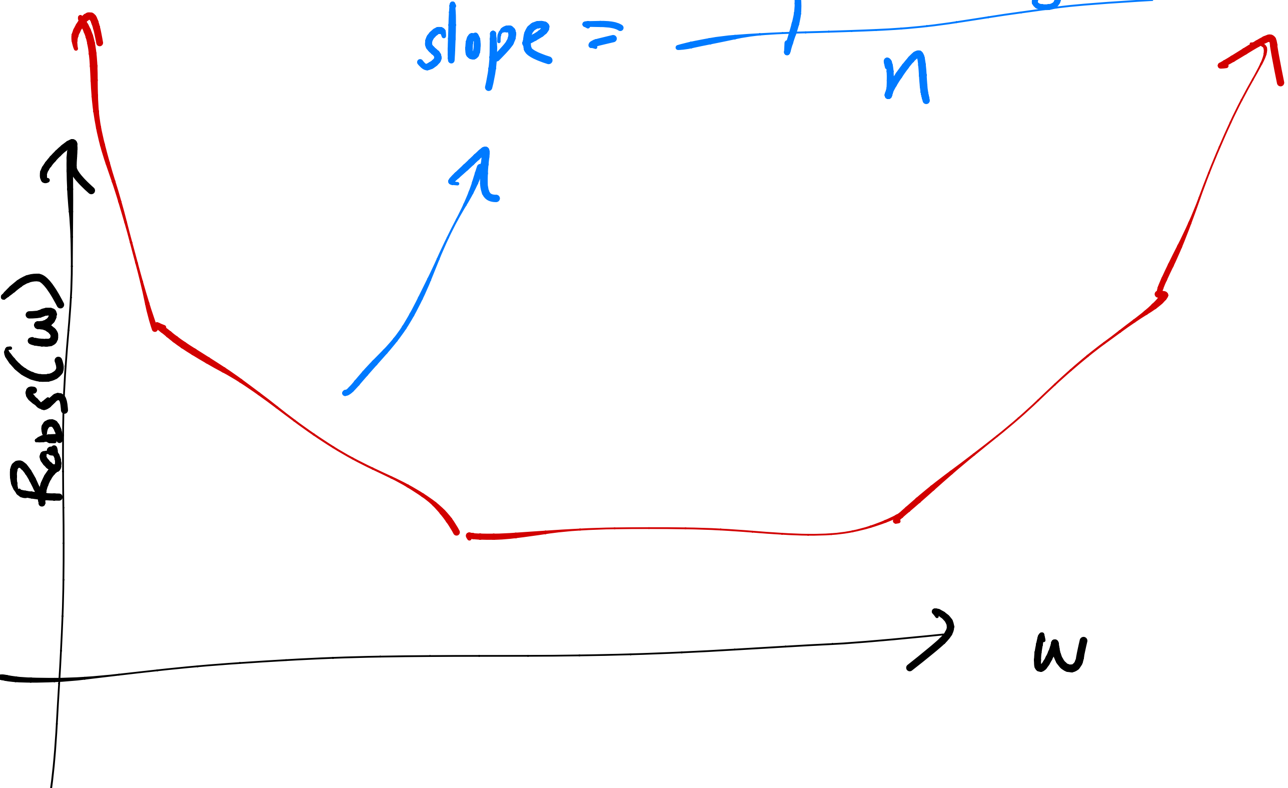
$$Z_{abs}(z_1) = \frac{z_5 - z_1 + z_4 - z_1 + \dots + z_1 - z_1}{5} = \frac{z_1 + \dots + z_5}{5} - z_1 = 12 - z_1$$

$$6 = \frac{3}{8}(z_1 - 2) + \frac{5}{8}(12 - z_1)$$

$\Rightarrow$ solve for z_1, you get $z_1 = 3$

slope of $R_{abs}(w)$

$$\text{slope} = \frac{\#\text{left} - \#\text{right}}{n}$$



wasn't on mock exam, but important

Review Ch. 1.4, Homework 2

Problem 3: Switcheroo

Consider the following datasets, both consisting of $n = 8$ points.

- "Old" dataset: $(\mathbf{3}, \mathbf{8}), (\mathbf{7}, \mathbf{2}), (x_3, y_3), \ldots, (x_8, y_8)$
- "New" dataset: $(\mathbf{3}, \mathbf{2}), (\mathbf{7}, \mathbf{8}), (x_3, y_3), \ldots, (x_8, y_8)$

Note that the only difference between the datasets is that the first two y-values have been swapped.

a) Which of the following quantities **are guaranteed to be different** between the old and new datasets? Select all that apply.

$$r = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma_x}\right)\left(\frac{y_i - \bar{y}}{\sigma_y}\right)$$

- [ ] The mean of the x-values, $\bar{x}$
- [ ] The mean of the y-values, $\bar{y}$
- [ ] The variance of the x-values, σ_x^2
- [ ] The variance of the y-values, σ_y^2
- [x] The correlation coefficient between the x-values and the y-values, r

b) Let m_{old} and m_{new} be the slopes of the regression lines fit to the old and new datasets, respectively. Given that $\sigma_x^2 = 50$, find the value of $\boxed{m_{\text{new}} - m_{\text{old}}}$. Your final answer should be a number with no variables.

Many formulas for optimal slope: easiest will be

$$w_1^* = \frac{\sum_{i=1}^{n} (x_i - \bar{x}) y_i}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

Only y_1, y_2 different in the 2 datasets

$$\rightarrow m_{new} = \frac{(3-\bar{x})2 + (7-\bar{x})8 + \sum_{i=3}^{8} (x_i - \bar{x}) y_i}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

shared in both

$$\rightarrow m_{old} = \frac{(3-\bar{x})8 + (7-\bar{x})2 + \sum_{i=3}^{8} (x_i - \bar{x}) y_i}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$m_{new} - m_{old}$

$$= \frac{(3-\bar{x})2 + (7-\bar{x})8 - \left((3-\bar{x})8 + (7-\bar{x})2\right)}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$= \frac{(3-\bar{x})\underbrace{(2-8)}_{-6} + (7-\bar{x})\underbrace{(8-2)}_{6}}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$= \frac{6\left(7-\cancel{\bar{x}}-3+\cancel{\bar{x}}\right)}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \frac{24}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

since variance $= \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = 50$,

$$\sum_{i=1}^{n}(x_i-\bar{x})^2 = 50n = 50\cdot 8 = 400$$

so $m_{new} - m_{old} = \frac{24}{400} = \frac{3}{50}$

$\vec{x}\cdot\vec{x} = (-4)^2 + 3^2 = 5^2 = 25$

$\vec{y}\cdot\vec{x} = -12 + 3c$

Problem 4: Variability

Let $\vec{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ and $y = \begin{bmatrix} 3 \\ c \end{bmatrix}$, where $c \in \mathbb{R}$ is a constant.

a) Fill in the blanks to complete the sentence:

The value of c that makes $|\vec{x}\cdot\vec{y}|$ as small as possible is ___(i)___; when using that value of c, $\vec{x}$ and $\vec{y}$ are ___(ii)___.

(i) 4

(ii) orthogonal (1-3 words)

b) Suppose the projection of $\vec{y}$ onto $\vec{x}$ is $\begin{bmatrix} -12/5 \\ 9/5 \end{bmatrix}$, for some value of c. What is the value of c?

Show your work, and circle your final answer, which should be a number with no variables.

projection of $\vec{y}$ onto $\vec{x}$

$$\begin{bmatrix} -12/5 \\ 9/5 \end{bmatrix} = \left(\frac{\vec{y}\cdot\vec{x}}{\vec{x}\cdot\vec{x}}\right)\vec{x} = \frac{-12+3c}{25}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$\frac{\cancel{25}^{\,5}}{\cancel{3}}\left(\frac{\cancel{9}^{\,3}}{\cancel{5}}\right) = \frac{-12+3c}{\cancel{25}}\cdot\cancel{3}$$

$15 = -12 + 3c$

$27 = 3c \Rightarrow \boxed{c = 9}$

As a refresher, $\vec{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 3 \\ c \end{bmatrix}$, where $c \in \mathbb{R}$ is a constant.

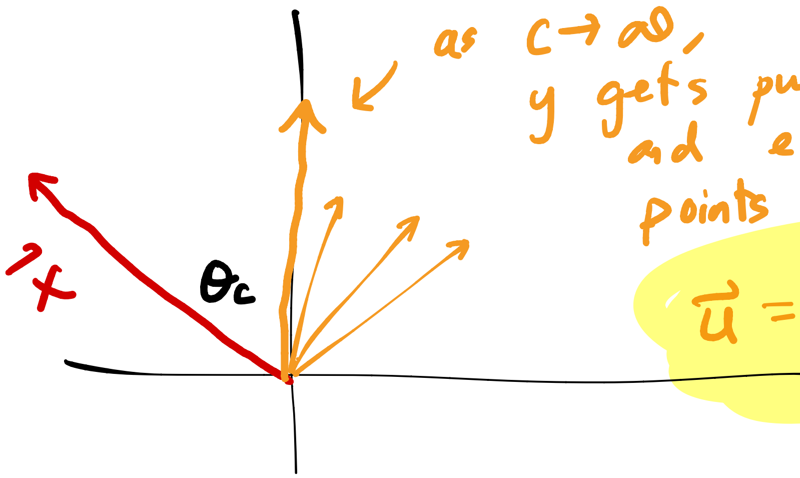
In the next two parts, suppose θ_c is the angle between $\vec{x}$ and $\vec{y}$. As c gets larger and larger, $\cos\theta_c$ gets closer and closer to L, i.e.

$$\lim_{c\to\infty} \cos\theta_c = L$$

c) What is the value of L?

- ○ $3/4$
- ○ $-3/4$
- ● $3/5$
- ○ $-3/5$
- ○ $4/5$
- ○ $-4/5$
- ○ None of these

d) $\cos^{-1}(L)$ is also equal to the angle between $\vec{x}$ and a particular unit vector, $\vec{u}$. Find $\vec{u}$ and explain your answer.
Hint: This is more of a conceptual question than a computational one.



as $c \to \infty$, y gets pulled up and eventually points in the direction

$$\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Part c)

Solution ① : Recognize that as $c\to\infty$, $\vec{y}$ starts to point in the direction of θ_c

$$\lim_{c\to\infty} \cos\theta_c = \frac{\vec{x}\cdot\begin{bmatrix}0\\1\end{bmatrix}}{\|\vec{x}\| \, \left\|\begin{bmatrix}0\\1\end{bmatrix}\right\|} = \frac{(-4)(0)+(3)(1)}{\sqrt{(-4)^2+3^2}\sqrt{0^2+1^2}} = \frac{3}{5}$$

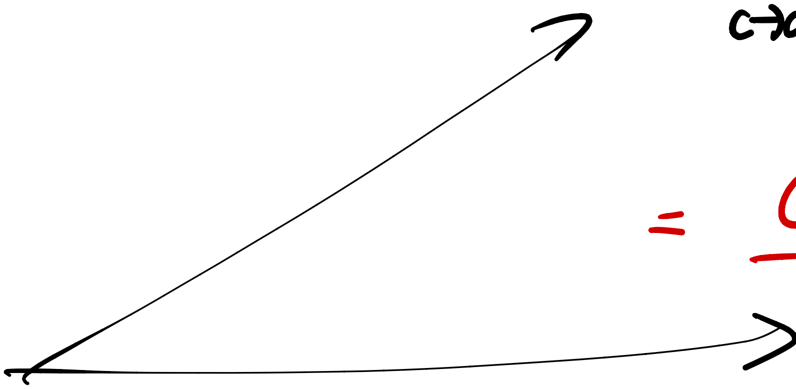
Solution (2):

$$\cos\theta_c = \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\,\|\vec{y}\|} = \frac{-12+3c}{5\sqrt{9+c^2}}$$

$$\vec{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 3 \\ c \end{bmatrix}$$

$$\lim_{c\to\infty} \cos\theta_c = \lim_{c\to\infty} \frac{-12+3c}{5\sqrt{9+c^2}}$$

$$= \lim_{c\to\infty} \frac{-\frac{12}{c}+3}{5\sqrt{\frac{9}{c^2}+1}}$$



$$= \frac{0+3}{5\sqrt{0+1}} = \frac{3}{5}$$

See next page

Problem 5: Well, It Depends

Let $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 6 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ \alpha \\ 2 \end{bmatrix}$, where $\alpha \in \mathbb{R}$ is a constant.

$\vec{u}, \vec{v}, \vec{w}$ lin. ind.

In parts **a)**, **b)**, **c)**, and **d)**, suppose $\boxed{\alpha = 3}$. Fill in the blanks to complete each sentence.

a) The two vectors $\{\vec{w}, \vec{u} - \vec{w}\}$ are ___(i)___, and span a ___(ii)___-dimensional subspace of $\mathbb{R}^4$.

(i) ● linearly independent ○ linearly dependent

(ii) ○ 1 ● 2 ○ 3 ○ 4

b) The two vectors $\{\vec{v}, 2\vec{v}\}$ are ___(i)___, and span a ___(ii)___-dimensional subspace of $\mathbb{R}^4$.

(i) ○ linearly independent ● linearly dependent

(ii) ● 1 ○ 2 ○ 3 ○ 4

c) The three vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ are ___(i)___, and span a ___(ii)___-dimensional subspace of $\mathbb{R}^4$.

(i) ● linearly independent ○ linearly dependent

(ii) ○ 1 ○ 2 ● 3 ○ 4

nothing new

d) The four vectors $\{\vec{u}, \vec{v}, \vec{w}, \vec{w} - \vec{u}\}$ are ___(i)___, and span a ___(ii)___-dimensional subspace of $\mathbb{R}^4$.

(i) ○ linearly independent ● linearly dependent

(ii) ○ 1 ○ 2 ● 3 ○ 4

$\vec{w}$ is lin comb of $\vec{u}, \vec{v}$

Now, suppose $\boxed{\alpha = -2}$. Fill in the blanks to complete each sentence.

e) The three vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ are ___(i)___, and span a ___(ii)___-dimensional subspace of $\mathbb{R}^4$.

(i) ○ linearly independent ● linearly dependent

(ii) ○ 1 ● 2 ○ 3 ○ 4

f) The four vectors $\{\vec{u}, \vec{v}, \vec{w}, \vec{w} - \vec{u}\}$ are ___(i)___, and span a ___(ii)___-dimensional subspace of $\mathbb{R}^4$.

(i) ○ linearly independent ● linearly dependent

(ii) ○ 1 ● 2 ○ 3 ○ 4

Dimension of subspace = # vectors

Which value of α makes $\vec{w}$ a linear combination of $\vec{u}, \vec{v}$?

$$\vec{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 4 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 6 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 1 \\ -1 \\ \alpha \\ 2 \end{bmatrix}$$

If $\vec{w}$ is a linear combination of $\vec{u}, \vec{v}$, then there exists a, b such that

$$a\vec{u} + b\vec{v} = \vec{w}$$

same eq'n

$$\begin{aligned} 2a + 3b &= 1 \\ b &= -1 \\ -a &= \alpha \\ 4a + 6b &= 2 \end{aligned}$$

implies $a = -\alpha$, $b = -1$, need to check first 2 eq'ns

$$2(-\alpha) + 3(-1) = 1$$

$$-2\alpha = 4$$

$$\alpha = -2$$

$$4(2) + 6(-1) = 2$$

so if $\alpha = -2$, $\vec{w}$ is lin comb of $\vec{u}, \vec{v}$, otherwise not

set of vectors that
① span all of S
② are linearly independent

Problem 6: Covering All the Bases

a) Find a basis for the following subspace of $\mathbb{R}^3$:

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid 6x - y + z = 0 \right\}$$

equation of a plane $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

need to find two vectors that
① satisfy $6x-y+z=0$
② aren't collinear

not the only ans!

$$\left\{ \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

dimension of subspace = # vectors in every basis for that subspace

b) The equations $6x - y + z = 0$ and $4x + 4y = 0$ intersect in a line in $\mathbb{R}^3$. Find the equation of this line in **parametric** form.

both are planes!

could try and solve

$$6x-y+z=0$$
$$4x+4y=0$$
$$\vdots$$

Easy solution: just find a vector in both planes, like $\begin{bmatrix} 1 \\ -1 \\ -7 \end{bmatrix}$

so, $L = t\begin{bmatrix} 1 \\ -1 \\ -7 \end{bmatrix}, \ t \in \mathbb{R}$

Problem 7: Catchy

Suppose $\vec{x} \in \mathbb{R}^n$. Prove that the L_1 norm of a vector is less than or equal to $\sqrt{n}$ times the L_2 norm of a vector, i.e.

$$\|\vec{x}\|_1 \leq \sqrt{n}\|\vec{x}\|_2$$

Hint: Use the Cauchy-Schwarz inequality, which states that for any two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|_2\|\vec{v}\|_2$. Most of your job is to choose the right vectors $\vec{u}$ and $\vec{v}$ to apply the Cauchy-Schwarz inequality to.

$$\|\vec{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \qquad \|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

After experimenting, try Cauchy-Schwarz ("C-S") with

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_n| \end{bmatrix}$$

C-S says $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|_2 \|\vec{v}\|_2$

$$(1)|x_1| + (1)|x_2| + \dots + (1)|x_n| \leq \sqrt{\underbrace{1^2 + \dots + 1^2}_{n}} \sqrt{\underbrace{|x_1|^2 + \dots + |x_n|^2}_{x_1^2 + \dots + x_n^2}}$$

$$\sum_{i=1}^{n} |x_i| \leq \sqrt{n} \|\vec{x}\|_2$$

$$\|\vec{x}\|_1 \leq \sqrt{n} \|\vec{x}\|_2$$ as required

Note: With this problem, the mock exam *may* be a bit longer than the real exam, but we're still including this problem here for extra practice.