Final Exam

EECS 245, Spring 2026 at the University of Michigan

Name: ______________________________

uniqname: ______________________________

UMID: ______________________________

Room: ◯ 1018 DOW (8-10AM) ◯ 1018 DOW (extended time)

Instructions

- This exam consists of 14 problems, worth a total of 130 points, spread across 14 pages (7 sheets of paper). **All problems count towards your Final Exam score; certain problems also count towards your Midterm 1 or Midterm 2 redemption scores.**
- You have 120 minutes to complete this exam, unless you have extended-time accommodations through SSD.
- Write your uniqname in the top right corner of every page in the space provided.
- For free response problems, show your work unless otherwise specified, and write your final answer in the box provided.
- For multiple choice problems, completely fill in bubbles and square boxes; if we cannot tell which option(s) you selected, you may lose points.
  ◯ A bubble means select one choice.
  ☐ A square box means select all that apply.
- You may refer to **3** two-sided handwritten notes sheets. No other resources or technology are allowed.

You are to abide by the University of Michigan/Engineering Honor Code. To receive a grade, please sign below.

I have neither given nor received aid on this exam, nor have I concealed any violations of the Honor Code.

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Problem 1 (14 pts) **Counts towards Midterm 1 redemption score**

Suppose we'd like to find the optimal constant parameter, w^*, for the constant model $h(x_i) = w$, using the following dataset of $n = 5$ values:

$$1, \quad 1, \quad 4, \quad 9, \quad 25$$

In each part, find the value of w^* that minimizes the given $R(w)$. Show your work in the space provided, and write your final answer in the bottom-right corner of the box. Your answers should be numbers with no variables. *Note: There is no need to use calculus here.*

a) (4 pts) $R(w) = \frac{1}{5}\sum_{i=1}^{5}(y_i - w)^2$

$w^* =$

b) (4 pts) $R(w) = \frac{1}{5}\sum_{i=1}^{5}(\sqrt{y_i} - w)^2$

$w^* =$

c) (4 pts) $R(w) = \frac{1}{5}\sum_{i=1}^{5}(y_i - \sqrt{w})^2$

$w^* =$

d) (2 pts) Which answer from above is also the minimizer of $R(w) = \sqrt{\frac{1}{5}\sum_{i=1}^{5}(y_i - w)^2}$?

○ Answer from part (a) ○ Answer from part (b) ○ Answer from part (c) ○ None

Problem 2 (9 pts) **Counts towards Midterm 1 redemption score**

Suppose we fit a simple linear regression model to a dataset of n points, $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, by minimizing mean squared error. Let $\bar{x}$ and $\bar{y}$ be the means of the x-values and y-values, respectively, and suppose the standard deviations σ_x and σ_y are both positive. Let

$$h(x_i) = w_0^* + w_1^* x_i$$

be the best simple linear regression line for the original dataset.

Now, we create a new dataset of $n+1$ points by starting with the original dataset and adding one new point,

$$(x_{n+1}, y_{n+1}) = (\bar{x}, c)$$

where c is a constant. Let

$$h_{\text{new}}(x_i) = w_0' + w_1' x_i$$

be the best simple linear regression line for the new dataset.

a) (6 pts) Prove that $w_1' = w_1^*$, i.e. that the new slope is the same as the old slope, no matter what c is. *Hint: Start with any of the formulas for the optimal slope that involve summations in the numerator and denominator, and separate the sums.*

b) (3 pts) Which of the following expressions is equal to $w_0' - w_0^*$, the difference between the new intercept and the old intercept?

○ $\frac{\bar{y} - c}{n}$ ○ $\frac{\bar{y} - c}{n+1}$ ○ $\frac{c - \bar{y}}{n}$ ○ $\frac{c - \bar{y}}{n+1}$ ○ None of these

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Problem 3 (10 pts) **Counts towards Midterm 1 redemption score**

Let $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix}$, and suppose $\vec{y} \in \mathbb{R}^3$ is such that

the projection of $\vec{x}$ onto $\vec{y}$ is $\vec{0}$ and that $\vec{y} \cdot \vec{y} = \vec{y} \cdot \vec{z} = 45$.

a) (4 pts) Find the projection of $\vec{z}$ onto $\vec{x}$. Show your work, and write your final answer in the box provided. Give your answer as a vector with no variables.

projection of $\vec{z}$ onto $\vec{x}$ = ☐

b) (6 pts) Write $\vec{z}$ as a linear combination of $\vec{x}$ and $\vec{y}$. Show your work, and fill in each box with a number with no variables. *Hint: What is the relationship between $\vec{x}$ and $\vec{y}$?*

$\vec{z} =$ ☐ $\vec{x} +$ ☐ $\vec{y}$

Problem 4 (5 pts) **Counts towards Midterm 1 redemption score**

Suppose $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 + x_2 + 2x_3 = 0 \text{ and } x_3 = x_4 \right\}$. State one basis for S. Your answer should be a list of vectors with no variables.

one basis for $S =$

Problem 5 (4 pts) **Counts towards Midterm 2 redemption score**

Suppose A is a 7×12 matrix. Fill in each blank with an integer with no variables.

(i) (2 pts) What is the minimum possible value of dim(nullsp(A))?

(ii) (2 pts) What is the maximum possible value of dim(nullsp(A))?

Problem 6 (6 pts) **Counts towards Midterm 2 redemption score**

Find the area enclosed by the polygon with vertices $(0, 0)$, $(4, 6)$, $(1, 8)$, and $(-3, 2)$. Show your work, and write your answer in the box provided.

area =

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Problem 7 (12 pts) **Counts towards Midterm 2 redemption score**

Suppose X is an $n \times d$ matrix with linearly independent columns, $d < n$, and $\vec{y} \in \mathbb{R}^n$.

Furthermore, suppose P is the matrix that projects vectors in $\mathbb{R}^n$ onto colsp(X), and $\vec{p} = P\vec{y}$ is the projection of $\vec{y}$ onto colsp(X).

Finally, let Q be an $n \times n$ orthogonal matrix.

a) (4 pts)

(i) (2 pts) What is det(P)? ○ -1 ○ 0 ○ 1 ○ -1 or 1 ○ None of these

(ii) (2 pts) What is det(Q)? ○ -1 ○ 0 ○ 1 ○ -1 or 1 ○ None of these

b) (2 pts) Which of the following vectors is orthogonal to colsp(X)?

○ $\vec{y}$ ○ $P\vec{y}$ ○ $Q\vec{y}$ ○ $(I - P)\vec{y}$ ○ $(I - Q)\vec{y}$ ○ None of these

c) (6 pts) Prove that the projection of $Q\vec{y}$ onto colsp(QX) is $Q\vec{p}$. *Hint: Start by showing that the matrix that projects vectors in* $\mathbb{R}^n$ *onto colsp*(QX) *is* QPQ^T.

Problem 8 (12 pts) **Counts towards Midterm 2 redemption score**

Suppose we'd like to fit a multiple linear regression model to predict $\texttt{cost}_i$, the cost in dollars of parking in an Ann Arbor parking garage, using $\texttt{hours}_i$, the number of hours parked.

For each row i, the corresponding augmented feature vector is $\text{Aug}(\vec{x}_i) = \begin{bmatrix} 1 & \texttt{hours}_i & \max(0, \texttt{hours}_i - 2) \end{bmatrix}^T$ so the model is of the form

$$h(\vec{x}_i) = w_0 + w_1 \texttt{hours}_i + w_2 \max(0, \texttt{hours}_i - 2)$$

The model is fit by minimizing mean squared error.

a) (4 pts) Suppose the dataset has four rows, and the number of hours parked in those rows is 3, 0, 5, and 1, respectively. Write the first four rows of the design matrix X. Your answer should be a matrix with four rows and no variables.

$X =$

b) (2 pts) Give a one-sentence English explanation of the meaning of w_2.

c) (6 pts) Once again, suppose the dataset has four rows. In each of the following subparts, we provide the number of hours parked in the dataset. Find the rank of the design matrix X in each case. Fill in each blank with an integer with no variables.

(i) (2 pts) 3, 0, 5, and 1 $\text{rank}(X) =$

(ii) (2 pts) 2, 0, 2, and 1 $\text{rank}(X) =$

(iii) (2 pts) 3, 4, 5, and 6 $\text{rank}(X) =$

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Problem 9 (9 pts)

Counts towards Midterm 2 redemption score

Let $\vec{a} \in \mathbb{R}^2$ and let

$$f(\vec{x}) = \log(\vec{a} \cdot \vec{x})$$

for all vectors $\vec{x}$ such that $\vec{a} \cdot \vec{x} > 0$; if $\vec{a} \cdot \vec{x} \leq 0$, then $f(\vec{x})$ is undefined. Suppose that

$$\nabla f\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix}$$

a) (3 pts) Which of the following could be $\vec{a}$? **Select all** that apply.

☐ $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ ☐ $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ☐ $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ ☐ $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$

b) (4 pts) Suppose we use gradient descent to minimize $f(\vec{x})$ using an initial guess of $\vec{x}^{(0)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and a learning rate of $\alpha = 1/2$. Find $\vec{x}^{(1)}$. Show your work, and write your answer in the box provided. Your answer should be a vector with no variables.

$\vec{x}^{(1)} =$

c) (2 pts) This part is unrelated to the previous parts.

Suppose $g : \mathbb{R} \to \mathbb{R}$. True or false: if g has a global minimum and no local maxima, it must be convex.

◯ True ◯ False

Problem 10 (12 pts)

Let $A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$.

a) (8 pts) Find all eigenvalues and eigenvectors of A. Show your work, and organize your answers as follows:

- Put the larger eigenvalue in λ_1, and a corresponding eigenvector in $\vec{v}_1$.
- Put the smaller eigenvalue in λ_2, and a corresponding eigenvector in $\vec{v}_2$.

$\lambda_1 =$ $\vec{v}_1 =$ $\lambda_2 =$ $\vec{v}_2 =$

b) (4 pts) True or false: for all integer values of k, the matrix $B = \begin{bmatrix} 2 & 4 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & k \end{bmatrix}$ is diagonalizable.

◯ True ◯ False

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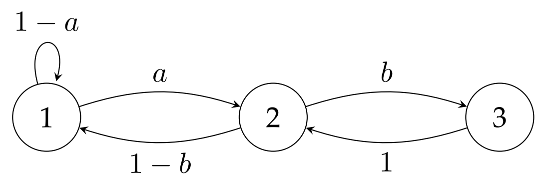
Problem 11 (10 pts)

The state diagram below describes a Markov chain with three states. a and b are both constants between 0 and 1.



Suppose that in the long run, $\frac{25}{60}$ of the time is spent in state 1, $\frac{21}{60}$ of the time is spent in state 2, and $\frac{14}{60}$ of the time is spent in state 3.

Find the values of a and b. Show your work, and write your final answers in the boxes provided. Your answers should be numbers with no variables.

$a =$ ☐ $b =$ ☐

Problem 12 (11 pts)

Suppose A is a 3×3 symmetric matrix with rank 2. The eigenspace corresponding to $\lambda = 9$ is the plane

$$2x - y + 2z = 0$$

Suppose $A = Q\Lambda Q^T$, where Q is an orthogonal matrix and Λ is a diagonal matrix with eigenvalues of A on the diagonal, **sorted** from largest to smallest.

a) (3 pts) Find Λ. Your answer should be a matrix with no variables.

$\Lambda =$

b) (8 pts) Consider the vector

$$\vec{v} = \begin{bmatrix} 2 \\ 9 \\ -2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Find $A\vec{v}$. Show your work, and write your final answer in the box provided. Your answer should be a vector with no variables. *Hint: What does the spectral theorem tell us?*

$A\vec{v} =$

Problem 13 (12 pts)

Let $\tilde{X}$ be a 4×2 centered matrix (i.e. in which each column has a mean of 0) with columns $\vec{a}$ and $\vec{b}$:

$$\tilde{X} = \begin{bmatrix} | & | \\ \vec{a} & \vec{b} \\ | & | \end{bmatrix}$$

Suppose $\tilde{X} = U\Sigma V^T$ is the singular value decomposition of $\tilde{X}$, $\vec{v}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$ is the first column of V, and $\sigma_1 = 10$ is the largest singular value.

a) (3 pts) How many possible vectors are there for $\vec{v}_2$, the second column of V?
◯ 1 ◯ 2 ◯ 3 ◯ 4 ◯ infinitely many $\vec{v}_2$'s are possible

b) (5 pts) Write $\vec{u}_1$, the first column of U, as a linear combination of the columns of $\tilde{X}$. Show your work, and fill in each box with a number with no variables.

$\vec{u}_1 =$ ☐ $\vec{a} +$ ☐ $\vec{b}$

c) (4 pts) Given the information above, what is the maximum possible variance of principal component 2? Give your answer as a number with no variables.

maximum possible variance of principal component 2 = ☐

Problem 14 (4 pts)

What is one topic you studied a lot for that was not on the Final Exam? **Blank answers will receive no credit!**

Congrats on completing the Final Exam for EECS 245! We'll really miss you; please stay in touch.

Feel free to draw us a picture about EECS 245 in the box below.