Midterm 2

EECS 245, Spring 2026 at the University of Michigan

Name: ______________________________

uniqname: ______________________________

UMID: ______________________________

Room: ◯ 1690 BBB (1-3PM) ◯ 1690 BBB (extended time)

Instructions

- This exam consists of 7 problems, worth 100 points, spread across 14 pages (7 sheets of paper).
- You have 120 minutes to complete this exam, unless you have extended-time accommodations through SSD.
- Write your uniqname in the top right corner of each page.
- For free response problems, **you must show all of your work**, and circle your final answer. We will not grade work that appears elsewhere, and you may lose points if your work is not shown.
- For multiple choice problems, completely fill in bubbles and square boxes; if we cannot tell which option(s) you selected, you may lose points.
  ◯ A bubble means that you should only select one choice.
  ☐ A square box means that you should select all that apply.
- You may refer to **two double-sided 8.5x11" handwritten notes sheets**. Other than that, you may not refer to any other resources or technology during the exam (no phones, watches, or calculators).

You are to abide by the University of Michigan/Engineering Honor Code. To receive a grade, please sign below to signify that you have kept the Honor Code pledge.

I have neither given nor received aid on this exam, nor have I concealed any violations of the Honor Code.

This page has been intentionally left blank.

uniqname: ______________________

Tip: Skim through the entire exam before starting to work on it.

Problem 1 (12 pts)

Suppose k is a real number. Let

$$A = \begin{bmatrix} 1 & k+1 \\ 1 & 2k+3 \end{bmatrix}$$

In each part, you are provided with information about A. **Your job is to find the value of k that satisfies the given condition.** Show your work in the space provided, and write your final answer in the bottom-right corner of the box. Your answers should be numbers with no variables.

a) (4 pts) $\det(A) = 14$.

$k =$ ______

b) (4 pts) A is not invertible.

$k =$ ______

c) (4 pts) The bottom-right entry of A^{-1} is $1/4$.

$k =$ ______

Problem 2 (16 pts)

Suppose A is a 3×3 matrix whose null space is the plane

$$5x - y + 3z = 0$$

In other words, $\text{nullsp}(A) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid 5x - y + 3z = 0 \right\}.$

a) (3 pts) Determine the following values. Give your answers as integers with no variables.

$\text{rank}(A) =$ $\boxed{}$ $\quad \dim(\text{nullsp}(A)) =$ $\boxed{}$

b) (4 pts) State one basis for $\text{nullsp}(A)$. Your answer should be a list of vectors with no variables.

one basis for $\text{nullsp}(A) =$ $\boxed{}$

though there are infinitely many possible answers.

c) (3 pts) State one basis for $\text{colsp}(A^T)$, the row space of A. Your answer should be a list of vectors with no variables.

one basis for $\text{colsp}(A^T) =$ $\boxed{}$

Recall, A is a 3×3 matrix whose null space is the plane $5x - y + 3z = 0$.

d) (6 pts) Suppose that

$$A \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 30 \\ 0 \end{bmatrix}$$

Find A. Show your work, and $\boxed{\text{circle}}$ your final answer, which should be a matrix with no variables.

Problem 3 (12 pts)

Suppose A is an $n \times n$ matrix.

For each statement below, determine whether it is true or false. If true, prove that it is true. If false, give a counterexample or a short explanation.

a) (4 pts) If A is symmetric, then A^2 must be symmetric.

◯ True ◯ False

b) (4 pts) If A^2 is symmetric, then A must be symmetric.

◯ True ◯ False

c) (4 pts) If $\vec{x} \in \text{nullsp}(A^T)$ and $\vec{y} \in \text{colsp}(A)$, then $\vec{x}$ and $\vec{y}$ are orthogonal.

◯ True ◯ False

Problem 4 (14 pts)

Suppose X is a matrix such that

$$X^T X = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \qquad X X^T = \begin{bmatrix} 1 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

a) (3 pts) Fill in each blank with an integer with no variables.

X has [ ] rows, [ ] columns, and rank$(X) =$ [ ].

b) (4 pts) For each statement below, determine whether it is true or false.

(i) The columns of X are all orthogonal to each other.
◯ True ◯ False

(ii) The columns of X are orthonormal.
◯ True ◯ False

c) (7 pts) Suppose P is the matrix that projects onto the column space of X. In other words, for any $\vec{y}$ of the appropriate shape, $P\vec{y}$ is the projection of $\vec{y}$ onto colsp(X). **Find** P. Show your work, and circle your final answer, which should be a matrix with no variables.

Problem 5 (19 pts)

Suppose we're given a dataset with $n = 5$ rows, and we use it to fit a multiple linear regression model with two features and an intercept term.

$$h(\vec{x}_i) = w_0 + w_1 x_i^{(1)} + w_2 x_i^{(2)}$$

Let X be the corresponding 5×3 design matrix and $\vec{y} \in \mathbb{R}^5$ be the corresponding observation vector. Suppose the matrix P that projects onto the column space of X is

$$P = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

a) (4 pts) **In parts a) and b) only**, suppose the projection of $\vec{y}$ onto $\text{colsp}(X)$ is $\vec{p} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$. There are infinitely many such vectors $\vec{y}$. State one possible vector $\vec{y}$ **whose five components are all different**. Give your answer as a vector with no variables.

one possible vector $\vec{y} =$ [ ]

b) (3 pts) Let $\vec{y}$ and $\vec{p}$ be as defined in part (a). True or false: $X^T(\vec{p} - \vec{y}) = \vec{0}$.

○ True ○ False

For the rest of the problem, suppose that both $\vec{w}^* = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and $\vec{w}' = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ are both optimal parameter vectors that minimize mean squared error.

c) (4 pts) Which of these vectors are in $\text{nullsp}(X)$? **Select all** that apply.

☐ $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ ☐ $\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}$ ☐ $\begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}$

The information stated below, above part d), is the same as the information stated on the previous page. It's provided for your convenience.

Recall, X is a 5×3 design matrix for the model

$$h(\vec{x}_i) = w_0 + w_1 x_i^{(1)} + w_2 x_i^{(2)}$$

Additionally, $\vec{y} \in \mathbb{R}^5$ is an observation vector, both $\vec{w}^* = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and $\vec{w}' = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ are both optimal parameter vectors that minimize mean squared error, and the matrix P that projects onto the column space of X is

$$P = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

d) (8 pts) Find one possible design matrix X, consistent with all of the information above. Show your work, and circle your final answer, which should be a matrix with no variables.

Problem 6 (12 pts)

Suppose A is an $n \times d$ matrix and $\vec{x} \in \mathbb{R}^d$. Consider the function $f : \mathbb{R}^d \to \mathbb{R}$ given by

$$f(\vec{x}) = \|A\vec{x}\|$$

a) (2 pts) True or False: $f(\vec{x})$ is a linear transformation.
◯ True ◯ False

b) (10 pts) Find $\nabla f(\vec{x})$. Assume that $A\vec{x} \neq \vec{0}$. Show your work, and write your final answer in the bottom-right corner of the box. Your answer should be an expression in terms of A, $\vec{x}$, and/or constants. *Hint: Start by taking the gradient of* $\|A\vec{x}\|^2$, *then apply the chain rule.*

$\nabla f(\vec{x}) =$

Problem 7 (15 pts)

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(\vec{x}) = cx_1^2 + dx_2^2$$

where c and d are constants. We'd like to use gradient descent to minimize $f(\vec{x})$. For some values of c and d, and some initial guess $\vec{x}^{(0)}$ and learning rate/step size α, we find that

$$\vec{x}^{(1)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad \nabla f(\vec{x}^{(1)}) = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \qquad \vec{x}^{(2)} = \begin{bmatrix} 2.8 \\ 1.4 \end{bmatrix}$$

a) (5 pts) Find the value of α. Show your work, and write your final answer in the bottom-right corner of the box. Your answer should be a number with no variables.

$\alpha =$ ____

b) (5 pts) Find the value of d (**not** c). Show your work, and write your final answer in the bottom-right corner of the boxes. Your answer should be a number with no variables.

$d =$ ____

c) (5 pts) Your friend claims that gradient descent always converges to a minimum because each iteration moves in the direction of steepest decrease. Based on the information in this problem, is your friend correct? State "yes" or "no", and briefly explain your reasoning.

This page has been intentionally left blank. Feel free to use it for scratch work.

Congrats on finishing Midterm 2!

Feel free to draw us a picture about EECS 245 in the box below.