Final Exam

EECS 245, Winter 2026 at the University of Michigan

Name: ______________________________

uniqname: ______________________________

UMID: ______________________________

Room: ◯ 1013 DOW ◯ 1017 DOW ◯ 2725 BBB ◯ Other

Instructions

- This exam consists of 13 problems, worth a total of 130 points, spread across 14 pages (7 sheets of paper). **All problems count towards your Final Exam score; certain problems also count towards your Midterm 1 or Midterm 2 redemption scores.**
- You have 120 minutes to complete this exam, unless you have extended-time accommodations through SSD.
- Write your uniqname in the top right corner of every page in the space provided.
- For free response problems, you must show all of your work (unless otherwise specified), and circle your final answer. We will not grade work that appears elsewhere, and you may lose points if your work is not shown.
- For multiple choice problems, completely fill in bubbles and square boxes; if we cannot tell which option(s) you selected, you may lose points.
  ◯ A bubble means that you should only select one choice.
  ☐ A square box means you should select all that apply.
- You may refer to **3** two-sided handwritten notes sheets. Other than that, you may not refer to any other resources or technology during the exam (no phones, watches, or calculators).

You are to abide by the University of Michigan/Engineering Honor Code. To receive a grade, please sign below to signify that you have kept the Honor Code pledge.

I have neither given nor received aid on this exam, nor have I concealed any violations of the Honor Code.

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Problem 1 (12 pts) **Counts towards Midterm 1 redemption score**

Suppose we'd like to find the optimal constant prediction, w^*, for the constant model $h(x_i) = w$, given the following dataset of $n = 4$ values.

$$y_1 = 3, \quad y_2 = 6, \quad y_3 = 6, \quad y_4 = 13$$

In each part, choose from the options below.

$A = 3$

$B = \dfrac{4}{\frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{13}} \approx 5.37$

$C = 6$

$D = (3 \cdot 6 \cdot 6 \cdot 13)^{1/4} \approx 6.12$

$E = 7$

$F = \sqrt{\dfrac{3^2 + 6^2 + 6^2 + 13^2}{4}} \approx 7.90$

$G = 8$

$H = 13$

(i) (3 pts) What value of w^* minimizes $R(w) = \dfrac{1}{4}\displaystyle\sum_{i=1}^{4}(y_i - w)^2$?

○ A ○ B ○ C ○ D ○ E ○ F ○ G ○ H

(ii) (3 pts) What value of w^* minimizes $R(w) = \displaystyle\lim_{p \to \infty} \frac{1}{4}\sum_{i=1}^{4} |y_i - w|^p$?

○ A ○ B ○ C ○ D ○ E ○ F ○ G ○ H

(iii) (3 pts) What value of w^* minimizes $R(w) = \dfrac{1}{4}\displaystyle\sum_{i=1}^{4}(\log(y_i) - \log(w))^2$?

○ A ○ B ○ C ○ D ○ E ○ F ○ G ○ H

(iv) (3 pts) The slope of the graph of $R(w) = \dfrac{1}{4}\displaystyle\sum_{i=1}^{4} |y_i - w|$ at $w = \alpha$ is $-1/2$. Among the options above, which could be α?

○ A ○ B ○ C ○ D ○ E ○ F ○ G ○ H

Problem 2 (13 pts) **Counts towards Midterm 1 redemption score**

Suppose a dataset of n points, $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, has the following properties:

mean of y-values $= \bar{y} = 11$, standard deviation of x-values $= \sigma_x = 2$, $\sigma_y = 6$

The simple linear regression line that minimizes mean squared error for predicting y_i from x_i is

$$h(x_i) = 15 - x_i$$

a) (3 pts) What is $\bar{x}$, the mean of the x-values? Give your answer as a number with no variables.

$\bar{x} =$ [ ]

Now, consider a new dataset, $(t_1, z_1), (t_2, z_2), \ldots, (t_n, z_n)$, defined by $t_i = 5 - x_i$ and $z_i = 2y_i - 1$.

Let $g(t_i) = \beta_0^* + \beta_1^* t_i$ be the best simple linear regression line for predicting z_i from t_i.

b) (6 pts) Find β_0^*, the intercept of the best simple linear regression line for predicting z_i from t_i. Show your work, and write your final answer in the box provided. Your answer should be a number with no variables.

$\beta_0^* =$ [ ]

c) (4 pts) Let M be the mean squared error of the model $h(x_i) = 15 - x_i$'s predictions on the dataset $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, and M' be the mean squared error of the model $g(t_i) = \beta_0^* + \beta_1^* t_i$'s predictions on the dataset $(t_1, z_1), (t_2, z_2), \ldots, (t_n, z_n)$.

What is the value of the fraction $\frac{M}{M'}$? *If it's not clear, M' is on the denominator.*

○ 1/5 ○ 1/4 ○ 1/2 ○ 1 ○ 2 ○ 4 ○ 5 ○ Impossible to tell

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Problem 3 (9 pts)

Counts towards Midterm 1 redemption score

a) (5 pts) Suppose $\vec{a} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ and that $\vec{b}$ is another vector in $\mathbb{R}^3$ such that:

- $\vec{a}$ and $\vec{b}$ are orthogonal, and
- the plane spanned by $\vec{a}$ and $\vec{b}$ is

$$4x - 2y + z = 0$$

There are infinitely many possible vectors $\vec{b}$ that satisfy the given conditions. State **one** of them. Show your work, and write your final answer in the box provided. Your answer should be a vector with no variables.

one possible $\vec{b} =$

b) (4 pts) This part is unrelated to the previous part. Suppose $\vec{u}, \vec{v} \in \mathbb{R}^n$, and that:

- $\vec{u}$ is a unit vector,
- $\cos(\theta) = 2/3$, where θ is the angle between $\vec{u}$ and $\vec{v}$,
- the projection of $\vec{v}$ onto $\vec{u}$ is $6\vec{u}$.

What is the value of $\|\vec{v}\|$?

○ 1 ○ 3 ○ 4 ○ 6 ○ 9

Problem 4 (4 pts) **Counts towards Midterm 1 redemption score**

Let

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \in \mathbb{R}^6 : x_1 + x_2 + x_3 = 0 \text{ and } x_4 = x_5 \right\}$$

Find $\dim(S)$. Give your answer as an integer with no variables.

$\dim(S) =$ ☐

Problem 5 (11 pts) **Counts towards Midterm 2 redemption score**

Suppose A is a 6×5 matrix such that

$$\text{nullsp}(A) = \text{span}\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \right)$$

a) (4 pts) Find $\text{rank}(A)$ and $\dim(\text{nullsp}(A^T))$. Give your answers as integers with no variables.

$\text{rank}(A) =$ ☐ $\dim(\text{nullsp}(A^T)) =$ ☐

b) (3 pts) Which of the following **could NOT** be the first row of A?

◯ $\begin{bmatrix} 2 & 2 & -2 & 3 & -3 \end{bmatrix}$ ◯ $\begin{bmatrix} 1 & 1 & -1 & 4 & -4 \end{bmatrix}$ ◯ $\begin{bmatrix} 2 & 0 & -2 & 5 & -5 \end{bmatrix}$ ◯ $\begin{bmatrix} 3 & 3 & -3 & -2 & 2 \end{bmatrix}$

c) (4 pts) Let $\vec{a}^{(1)}, \vec{a}^{(2)}, \vec{a}^{(3)}, \vec{a}^{(4)}, \vec{a}^{(5)} \in \mathbb{R}^6$ be the columns of A.

Below, select **one possible set** of columns of A that form a basis for $\text{colsp}(A)$. You should select the fewest possible number of columns needed to span $\text{colsp}(A)$.

Column	Include in your basis?
$\vec{a}^{(1)}$	☐
$\vec{a}^{(2)}$	☐
$\vec{a}^{(3)}$	☐
$\vec{a}^{(4)}$	☐
$\vec{a}^{(5)}$	☐

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Problem 6 (12 pts) **Counts towards Midterm 2 redemption score**

Suppose X is an $n \times 3$ matrix, where $n > 2$, with columns $\vec{x}^{(1)}$, $\vec{x}^{(2)}$, and $\vec{x}^{(3)}$. Furthermore, suppose that $X = QR$, where

$$Q = \begin{bmatrix} | & | \\ \vec{q}^{(1)} & \vec{q}^{(2)} \\ | & | \end{bmatrix}$$

is an $n \times 2$ matrix with orthonormal columns, and

$$R = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Lastly, suppose $\vec{y} \in \mathbb{R}^n$ and $Q^T \vec{y} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$.

a) (6 pts) Let $\vec{p}$ be the projection of $\vec{y}$ onto colsp(X). Write $\vec{p}$ as a linear combination of the columns of X. Fill in each box with a number with no variables. If there are multiple correct answers, you only need to provide one.

$$\vec{p} = \boxed{}\,\vec{x}^{(1)} + \boxed{}\,\vec{x}^{(2)} + \boxed{}\,\vec{x}^{(3)}$$

b) (6 pts) Let $\vec{w}^*$ be a minimizer of

$$R_{\text{sq}}(w) = \frac{1}{n}\|\vec{y} - X\vec{w}\|^2$$

Fill in the blanks to describe the set of all possible values of $\vec{w}^*$. Each blank should contain a vector with no variables.

$$\text{set of all possible } \vec{w}^* = \left\{ \boxed{} + t \boxed{} : t \in \mathbb{R} \right\}.$$

Problem 7 (8 pts) **Counts towards Midterm 2 redemption score**

Suppose we'd like to fit a multiple linear regression model **without an intercept term** to predict an apartment's monthly rent (in hundreds of dollars) using various features.

For apartment i, the corresponding feature vector is $\vec{x}_i = \begin{bmatrix}\text{bedrooms}_i & K_i & C_i & N_i\end{bmatrix}^T$, where bedrooms$_i$ is the number of bedrooms in apartment i, and K_i, C_i, and N_i are one hot encoded features for the Kerrytown, Central Campus, and North Campus neighborhoods, respectively.

The model is fit by minimizing mean squared error. **All rows of the dataset are shown to the right.** The model's predictions, $h(x_i)$, are shown, along with the true rents, y_i. Several values are missing.

bedrooms$_i$	neighborhood$_i$	y_i	$h(x_i)$
4	K	17	(i)
1	C	(ii)	9
3	C	15	13
2	C	10	11
1	N	9	(iii)
4	N	13	(iv)

For instance, the first row of the design matrix is $\begin{bmatrix}4 & 1 & 0 & 0\end{bmatrix}$.

Find all four missing values in the table. Show your work, and write your final answers in the boxes provided. Your answers should be integers with no variables. *Hint: Think about orthogonality.*

$(i) =$ ______ $(ii) =$ ______ $(iii) =$ ______ $(iv) =$ ______

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Problem 8 (9 pts) **Counts towards Midterm 2 redemption score**

Consider the function $g : \mathbb{R}^3 \to \mathbb{R}$. We'd like to minimize g using gradient descent.

a) (6 pts) Suppose two separate runs of gradient descent are started from **the same initial guess** $\vec{x}^{(0)}$, but with different learning rates (step sizes), α.

If $\alpha = 1/2$, then $\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and if $\alpha = 1/4$, then $\vec{x}^{(1)} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$.

Find $\nabla g(\vec{x}^{(0)})$, the gradient of g at $\vec{x}^{(0)}$. Show your work, and write your final answer in the box provided. Your answer should be a vector with no variables.

$\nabla g(\vec{x}^{(0)}) =$ [ ]

Now let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(\vec{x}) = (x_1 + 2x_2 - 6)^2 + \|\vec{x}\|^2$$

b) (3 pts) Suppose

$$\nabla f(\vec{x}) = M \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

for some 2×3 matrix M. Which of the following matrices is M?

○ $\begin{bmatrix} 2 & 2 & -6 \\ 2 & 5 & -12 \end{bmatrix}$ ○ $\begin{bmatrix} 4 & 2 & -12 \\ 2 & 10 & -24 \end{bmatrix}$ ○ $\begin{bmatrix} 4 & 4 & -12 \\ 4 & 10 & -24 \end{bmatrix}$ ○ $\begin{bmatrix} 4 & 4 & 12 \\ 4 & 10 & 24 \end{bmatrix}$ ○ $\begin{bmatrix} 4 & 4 & -12 \\ 2 & 6 & -12 \end{bmatrix}$

Problem 9 (12 pts)

Consider the matrix $A = \begin{bmatrix} 2 & 3 \\ -4 & k \end{bmatrix}$ where $k \in \mathbb{R}$ is some unknown constant.

a) (3 pts) Suppose $\lambda_1 = 0$ is an eigenvalue of A. Find the value of k. Give your answer as a number with no variables.

$k =$ ☐

b) (4 pts) Suppose $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A. Find the value of k. Give your answer as a number with no variables.

$k =$ ☐

c) (5 pts) Suppose $\lambda_1 = 3$ is an eigenvalue of A. Find λ_2, the **other eigenvalue** of A. Show your work, and write your final answer in the box provided. Give your answer as a number with no variables.

$\lambda_2 =$ ☐

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Problem 10 (14 pts)

The state diagram below describes a Markov chain with four states.

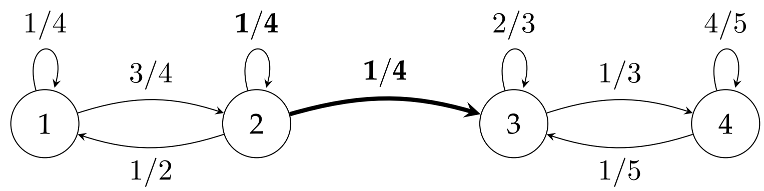
a) (4 pts) Find the adjacency matrix A for this Markov chain.

$A =$ [ ]

b) (6 pts) Suppose the chain starts in **state 1**. Fill each box with the **long-run fraction** of time spent in each state. Your answers should be numbers with no variables, and should sum to 1.

State 1: [ ] State 2: [ ] State 3: [ ] State 4: [ ]

Now, consider a **modified** version of the Markov chain. Changes have been emphasized in **bold**.



c) (4 pts) Consider the statement: "'If we start in ____, the long-run fraction of time spent in each state is the same as in the original chain."

Which of the following could be placed in the blank to make the statement true? **Select all** that apply.

☐ state 1 ☐ state 2 ☐ state 3 ☐ state 4 ☐ none of these are valid

Problem 11 (10 pts)

Let S be a 3×3 **symmetric** matrix with eigenvectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ corresponding to eigenvalues 5, 2, and -1, respectively. Assume that each $\vec{v}_i$ is a unit vector.

Suppose $\vec{x} \in \mathbb{R}^3$ and that

$$\vec{x} = 3\vec{v}_1 - 4\vec{v}_2 + \vec{v}_3$$

a) (6 pts) Write $S^2\vec{x}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. Fill in each box with a number with no variables.

$$S^2\vec{x} = \boxed{}\,\vec{v}_1 + \boxed{}\,\vec{v}_2 + \boxed{}\,\vec{v}_3$$

b) (4 pts) What is the value of $\|S\vec{x}\|^2$?

○ 24 ○ 26 ○ 218 ○ 290 ○ 5882 ○ Not enough information

Problem 12 (12 pts)

Suppose $\tilde{X}$ is an $n \times 2$ matrix whose columns are mean-centered (i.e. have a mean of 0). Furthermore, suppose

$$\tilde{X}^T \tilde{X} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

Note that $\tilde{X}^T \tilde{X}$ has eigenvalues of 7 and 2. Let $\tilde{X} = U \Sigma V^T$ be the singular value decomposition of $\tilde{X}$, and let $\vec{v}_1$ be the first column of V (not V^T).

a) (4 pts) What is $\vec{v}_1$? Give your answer as a vector with no variables. If there are multiple correct answers, you only need to provide one.

$\vec{v}_1 =$

b) (3 pts) Suppose the variance of the **second** principal component is $1/15$. What is n, the number of rows in $\tilde{X}$? Give your answer as a number with no variables.

$n =$

c) (5 pts) Suppose that $\vec{u}_2$ is the second column of U, corresponding to the singular value σ_2, in the singular value decomposition of $\tilde{X}$.
Prove that $\tilde{X}\vec{v}_1$ and $\sigma_2 \vec{u}_2$ are orthogonal. You do not need to re-prove any facts about the singular value decomposition, but you should state any facts you use.

Problem 13 (4 pts)

What is one topic you studied a lot for that wasn't on the Final Exam? **Blank answers will receive no credit!**

Congrats on completing the Final Exam for EECS 245! We'll really miss you; please stay in touch.

Feel free to draw us a picture about EECS 245 in the box below.

Did you notice any violations of the Honor Code during the exam? If so, share details with us here. We will keep your identity anonymous when investigating any cases.