Midterm 2

EECS 245, Winter 2026 at the University of Michigan

Name: ______________________________

uniqname: ______________________________

UMID: ______________________________

Room: ◯ 1670 BBB (big room) ◯ 1690 BBB ◯ Other

Instructions

- This exam consists of 8 problems, worth a total of 100 points, spread across 12 pages (6 sheets of paper).
- You have 120 minutes to complete this exam, unless you have extended-time accommodations through SSD.
- Write your uniqname in the top right corner of each page.
- For free response problems, **you must show all of your work**, and circle your final answer. We will not grade work that appears elsewhere, and you may lose points if your work is not shown.
- For multiple choice problems, completely fill in bubbles and square boxes; if we cannot tell which option(s) you selected, you may lose points.
  ◯ A bubble means that you should only select one choice.
  ☐ A square box means you should select all that apply.
- You may refer to **two double-sided 8.5x11" handwritten notes sheets**. Other than that, you may not refer to any other resources or technology during the exam (no phones, watches, or calculators).

You are to abide by the University of Michigan/Engineering Honor Code. To receive a grade, please sign below to signify that you have kept the Honor Code pledge.

I have neither given nor received aid on this exam, nor have I concealed any violations of the Honor Code.

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Tip: Skim through the entire exam before starting to work on it.

Problem 1 (12 pts)

Suppose k is a real number. Let

$$A = \begin{bmatrix} 3 & 2 \\ k & 4 \end{bmatrix}$$

In each part, you are provided with information about A. **Your job is to find the value of k that satisfies the given condition.** Show your work in the space provided, and write your final answer in the bottom-right corner of the box. Your answers should be numbers with no variables.

a) (4 pts) $\text{rank}(A) = 1$

$k =$ ☐

b) (4 pts) $\det(A) = 2$

$k =$ ☐

c) (4 pts) $A^{-1} = \begin{bmatrix} 1 & -1/2 \\ -1 & 3/4 \end{bmatrix}$

$k =$ ☐

Problem 2 (10 pts)

Suppose A is a matrix such that $A \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}$ and $\left\{ \begin{bmatrix} 1 \\ 4 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for nullsp(A).

Find one possible matrix A. Show your work, and circle your final answer, which should be a matrix with no variables.

Problem 3 (11 pts)

Suppose A and B are both **non-zero** 6×6 matrices, such that rank$(A) = 4$ and that every column of B is in nullsp(A).

a) (3 pts) Fill in the blanks: The third __(i)__ of A is __(ii)__ to the fourth __(iii)__ of B.

(i) ◯ column ◯ row

(ii) ◯ orthogonal ◯ parallel

(iii) ◯ column ◯ row

b) (4 pts) **Select all** possible values of rank(AB).

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

c) (4 pts) **Select all** possible values of rank(B).

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

Problem 4 (13 pts)

Suppose X is some $3 \times d$ matrix, for some integer d. Let

$$\vec{y} = \begin{bmatrix} 9 \\ -5 \\ 3 \end{bmatrix}$$

a) (5 pts) Which of the following **could** be the projection of $\vec{y}$ onto colsp(X)?
Select an answer, then briefly justify your answer in the space provided using properties of projections. Correct answers without justification may not receive full credit.

○ $\begin{bmatrix} 5 \\ -7 \\ 4 \end{bmatrix}$ ○ $\begin{bmatrix} 7 \\ -7 \\ 4 \end{bmatrix}$ ○ $\begin{bmatrix} 6 \\ -7 \\ 4 \end{bmatrix}$ ○ $\begin{bmatrix} 6 \\ -7 \\ 3 \end{bmatrix}$

In each of the remaining parts, identify whether the statement is True or False and justify your answer in the space provided. Correct answers without justification may not receive full credit.

b) (4 pts) If the projection of $\vec{y}$ onto colsp(X) is $\vec{y}$ itself, then rank(X) must be 3.
○ True ○ False

c) (4 pts) If rank$(X) = 3$, then the projection of $\vec{y}$ onto colsp(X) must be $\vec{y}$ itself.
○ True ○ False

Problem 5 (13 pts)

Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 4 \\ 3 & 1 & 0 & -7 & 4 \end{bmatrix}$$

a) (5 pts) Recall, a CR decomposition of an $n \times d$ matrix A is a product $A = CR$, where C is an $n \times r$ matrix with linearly independent columns and R is an $r \times d$ matrix with linearly independent rows, and $r = \text{rank}(A)$.

Provide a CR decomposition of A. Your answers should be matrices with no variables.

$C =$ [ ], $R =$ [ ]

b) (3 pts) Find $\dim(\text{nullsp}(A^T))$. Give your answer as an integer with no variables.

$\dim(\text{nullsp}(A^T)) =$ [ ]

c) (5 pts) Suppose we apply the Gram-Schmidt process to the **rows** of A, and place the resulting orthonormal vectors into the **rows** of a new matrix, Q.

Let P be the matrix that projects vectors in $\mathbb{R}^5$ onto $\text{colsp}(Q^T)$ (the row space of Q). In other words, if $\vec{y} \in \mathbb{R}^5$, then $P\vec{y}$ is the projection of $\vec{y}$ onto $\text{colsp}(Q^T)$.

Find an expression for P in terms of Q and Q^T. Show your work, and circle your final answer, which should be an expression in terms of Q and Q^T. Answers that aren't fully simplified will not be given credit.

$P =$ [ ]

Problem 6 (20 pts)

Suppose we'd like to fit a multiple linear regression model **without** an intercept term to **predict the number of fans in attendance at a Michigan football home game** given various features.

For each row in the dataset, the corresponding feature vector is $\vec{x}_i = \begin{bmatrix} \text{tempF}_i \\ \text{tempC}_i \\ \text{night}_i \\ \text{day}_i \end{bmatrix}$, where:

- tempF_i is the temperature, in degrees **Fahrenheit**, at kickoff for game i
- tempC_i is the temperature, in degrees **Celsius**, at kickoff for game i
- night_i is 1 if game i is a night game and 0 otherwise
- day_i is 0 if game i is a night game and 1 otherwise

Important: Note that

$$\text{tempC}_i = \frac{5}{9}(\text{tempF}_i - 32)$$

So, our model is of the form

$$h(\vec{x}_i) = w_1 \cdot \text{tempF}_i + w_2 \cdot \text{tempC}_i + w_3 \cdot \text{night}_i + w_4 \cdot \text{day}_i$$

We find optimal model parameters, $\vec{w}^* = \begin{bmatrix} w_1^* \\ w_2^* \\ w_3^* \\ w_4^* \end{bmatrix}$, by solving the normal equation.

a) (4 pts) The first two rows of the dataset have the following information:

- Game 1: 77 degrees Fahrenheit, 25 degrees Celsius, not night game, 102,111 fans
- Game 2: 59 degrees Fahrenheit, 15 degrees Celsius, night game, 101,982 fans

Write the first two rows of the design matrix, X. Your answer should be a matrix with two rows and no variables.

$X =$

Recall, our model is of the form

$$h(\vec{x}_i) = w_1 \cdot \text{tempF}_i + w_2 \cdot \text{tempC}_i + w_3 \cdot \text{night}_i + w_4 \cdot \text{day}_i$$

where $\text{tempC}_i = \frac{5}{9}(\text{tempF}_i - 32)$.

b) (4 pts) Suppose $\vec{w}'$ is one solution to the normal equation for this model. Which option describes the **complete set** of solutions to the normal equation?

◯ $\left\{ \vec{w}' + t \begin{bmatrix} 1 \\ -9/5 \\ 32 \\ -32 \end{bmatrix}, t \in \mathbb{R} \right\}$ ◯ $\left\{ \vec{w}' + t \begin{bmatrix} 1 \\ -9/5 \\ -32 \\ -32 \end{bmatrix}, t \in \mathbb{R} \right\}$

◯ $\left\{ \vec{w}' + t \begin{bmatrix} 1 \\ 9/5 \\ 32 \\ -32 \end{bmatrix}, t \in \mathbb{R} \right\}$ ◯ $\left\{ \vec{w}' + t \begin{bmatrix} 1 \\ 9/5 \\ 32 \\ -32 \end{bmatrix}, t \in \mathbb{R} \right\}$

c) (6 pts) First, assume $h(\vec{x}_i)$ is the model at the top of the page.

(i) What is the **largest possible** rank of the design matrix, X? (Note that we're asking about the full design matrix, not just its first two rows.)

largest possible value of rank(X) = ☐

(ii) True or False: The sum of the errors of the model's predictions is 0.

◯ True ◯ False

(iii) True or False: The sum of the errors of the model's predictions **on just the rows of the dataset corresponding to night games** is 0.

◯ True ◯ False

d) (6 pts) Now, suppose we remove the $\textbf{day}_i$ feature from our model, meaning our model is

$$h(\vec{x}_i) = w_1 \cdot \text{tempF}_i + w_2 \cdot \text{tempC}_i + w_3 \cdot \text{night}_i$$

(i) After removing the day column, what is the **largest possible** rank of the **new** design matrix?

largest possible value of rank(new design matrix) = ☐

(ii) True or False: The sum of the errors of the new model's predictions is 0.

◯ True ◯ False

(iii) True or False: The sum of the errors of the new model's predictions **on just the rows of the dataset corresponding to night games** is 0.

◯ True ◯ False

Problem 7 (10 pts)

a) (6 pts) Suppose A is a 4×4 matrix and $\vec{x} \in \mathbb{R}^4$. Furthermore, suppose that the gradient of the function $f(\vec{x}) = \vec{x}^T A\vec{x}$ is given by

$$\nabla f(\vec{x}) = \begin{bmatrix} 2x_1 \\ -15x_2 \\ 10x_3 \\ x_4 \end{bmatrix}$$

Find one possible matrix A. Your answer should be a 4×4 matrix with no variables.

$A =$

b) (4 pts) Suppose A is an $n \times n$ matrix, $\vec{b} \in \mathbb{R}^n$, and that $g : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$g(\vec{x}) = (\vec{b}^T A\vec{x})^2$$

Which of the following is $\nabla g(\vec{x})$, the gradient of $g(\vec{x})$?

- ◯ $2A^T\vec{b}$
- ◯ $(\vec{b}^T A\vec{x})A^T\vec{b}$
- ◯ $2(\vec{b}^T A\vec{x})\vec{b}$
- ◯ $2(\vec{b}^T A\vec{x})A^T\vec{b}$
- ◯ $2(\vec{b}^T A\vec{x})A^T\vec{x}$
- ◯ None of the above

Problem 8 (11 pts)

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Consider the function

$$f(\vec{x}) = (x_1 + x_2 - 4)^2$$

a) (3 pts) Fill in the blanks: The set of all vectors $\vec{x}^*$ that minimize $f(\vec{x})$ form a __(i)__ in $\mathbb{R}^2$. This set __(ii)__ a subspace of $\mathbb{R}^2$.

(i) ◯ point ◯ line ◯ plane

(ii) ◯ is not ◯ is

b) (8 pts) Suppose we use gradient descent to minimize $f(\vec{x})$ using an initial guess of $\vec{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Find the learning rate/step size α that will cause gradient descent to converge to a global minimum of $f(\vec{x})$ **in one iteration**, i.e. such that $\vec{x}^{(1)}$ is a minimizer of $f(\vec{x})$.

Show your work in the space provided, and write your final answer in the bottom-right corner of the box. Your answer should be a number with no variables.

$\alpha =$ ____

Congrats on finishing Midterm 2!

Feel free to draw us a picture about EECS 245 in the box below (or use it for scratch work).

Did you notice any violations of the Honor Code during the exam? If so, share details with us here. We will keep your identity anonymous when investigating any cases.