Homework 4: Projections, Spans, and Linear Independence

EECS 245, Fall 2025 at the University of Michigan

due Friday, September 26th, 2025 at 11:59PM Ann Arbor Time

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Homeworks are due to Gradescope by 11:59PM on the due date. See the **syllabus** for details on the slip day policy.

Homework will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain and justify your conclusions, using sound reasoning. Your goal should be to convince the reader of your assertions. If a question does not require explanation, it will be explicitly stated.

Before proceeding, make sure you're familiar with the **collaboration policy**.

Total Points: 6 + 7 + 11 + 7 + 12 + 6 + 8 = 57

Note that this homework has more problems than usual, but most of the problems are shorter than those in previous homeworks.

Problem 1: Warmup (6 pts)

Let $\vec{u} = \begin{bmatrix} k \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, where $k \in \mathbb{R}$ is some constant.

a) (1 pt) Find all values of k such that $\vec{u}$ and $\vec{v}$ are orthogonal.

b) (2 pts) Find all values of k such that $\text{span}(\{\vec{u}, \vec{v}\}) = \mathbb{R}^2$.

c) (3 pts) Find all values of k such that $\text{span}(\{\vec{u}, \vec{v}\})$ is a line in $\mathbb{R}^2$. Then, write the equation of that line, in both slope-intercept form ($y = mx + b$) and parametric form.

Problem 2: Projections (7 pts)

Suppose $\vec{u}, \vec{v} \in \mathbb{R}^n$. Let $\vec{p}$ be the projection of $\vec{u}$ onto $\vec{v}$, and let $\vec{e} = \vec{u} - \vec{p}$.

a) (2 pts) Which of the following vectors is $\vec{e}$ orthogonal to, and why? Select all that apply.

$$\vec{u}, \quad \vec{v}, \quad \vec{p}$$

(You don't need to rederive any results from Chapter 2.3, but we do want to hear your reasoning.)

b) (3 pts) $\text{span}(\{\vec{u}, \vec{v}\})$ is the set of all possible linear combinations of $\vec{u}$ and $\vec{v}$. Similarly, $\text{span}(\{\vec{e}, \vec{v}\})$ is the set of all possible linear combinations of $\vec{e}$ and $\vec{v}$.

Let's prove that $\text{span}(\{\vec{e}, \vec{v}\}) = \text{span}(\{\vec{u}, \vec{v}\})$, meaning that every vector you can create with $\vec{u}$ and $\vec{v}$ can also be created with $\vec{e}$ and $\vec{v}$, and vice versa.

Remember that the span of a set of vectors is a **set** too. To show that two sets A and B are equal, we need to show that every element of A is in B, and every element of B is in A.

Here, we'll show that

(i) if $\vec{x} \in \text{span}(\{\vec{u}, \vec{v}\})$, then $\vec{x} \in \text{span}(\{\vec{e}, \vec{v}\})$,

(ii) if $\vec{x} \in \text{span}(\{\vec{e}, \vec{v}\})$, then $\vec{x} \in \text{span}(\{\vec{u}, \vec{v}\})$.

We'll do **(i)** for you. If $\vec{x} \in \text{span}(\{\vec{u}, \vec{v}\})$, then

$$\vec{x} = a\vec{u} + b\vec{v}$$

for some scalars a and b. But, we know that $\vec{e} = \vec{u} - \vec{p}$, meaning that $\vec{u} = \vec{e} + \vec{p}$. This gives

$$\vec{x} = a(\vec{e} + \vec{p}) + b\vec{v} = a\vec{e} + a\vec{p} + b\vec{v}$$

But, $\vec{p}$ — the projection of $\vec{u}$ onto $\vec{v}$ — is a vector in the direction of $\vec{v}$, meaning that $\vec{p} = c\vec{v}$ for some scalar c. (Chapter 2.3 has the optimal value of c but it's not important in this proof.) Substituting $\vec{p} = c\vec{v}$ gives use

$$\vec{x} = a\vec{e} + a\vec{p} + b\vec{v} = a\vec{e} + a(c\vec{v}) + b\vec{v} = a\vec{e} + (ac + b)\vec{v}$$

This last expression, $a\vec{e} + (ac + b)\vec{v}$, is a linear combination of $\vec{e}$ and $\vec{v}$, meaning that $\vec{x} \in \text{span}(\{\vec{e}, \vec{v}\})$, as required.

Your turn: complete **(ii)** by showing that if $\vec{x} \in \text{span}(\{\vec{e}, \vec{v}\})$, then $\vec{x} \in \text{span}(\{\vec{u}, \vec{v}\})$.

c) (2 pts) To recap, the point of the previous part was to show that any vector that can be created with $\vec{u}$ and $\vec{v}$ can also be created with $\vec{e}$ and $\vec{v}$.

Using what you learned in Chapter 2.3 and Lab 4, explain why we'd rather write some new vector $\vec{b}$ as a linear combination of $\vec{e}$ and $\vec{v}$, rather than $\vec{u}$ and $\vec{v}$.

Problem 3: Lines and Planes (11 pts)

As we saw in Chapter 2.4, the span of two linearly independent vectors in $\mathbb{R}^n$ is a 2-dimensional subspace of $\mathbb{R}^n$, which we call a plane when working with vectors from $\mathbb{R}^3$. In this problem, we will build your understanding of lines and planes in $\mathbb{R}^3$.

To help you visualize lines and planes, consult:

- **The supplemental Jupyter Notebook** we've created for Homework 4, which can either be found **here** on DataHub, or **here** in the course GitHub repository.
- **The Lab 4 solutions**, which can be found **here**.
- Chapter 2.5 of the notes, which are being actively developed.

a) (3 pts) Consider the plane $7x - 3y + 4z = 0$. This is a plane written in standard form, $ax + by + cz + d = 0$.

Find two vectors that lie in this plane, and use those vectors to write the plane in parametric form. (There are infinitely many possible answers, since the parametric form of a line, or plane, or subspace in general is not unique.)

b) (8 pts) Consider the linearly independent vectors

$$\vec{v}_1 = \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

(i) In standard form, find the equation of the plane spanned by $\vec{v}_1$ and $\vec{v}_2$.

Hint: Use the cross product to find the values of a, b, and c in $ax + by + cz + d = 0$, and you know what d must be by the definition of the span of a set of vectors. Refer to Lab 4.

(ii) In standard form, find the equation of the plane spanned by $\vec{v}_1$ and $\vec{v}_3$.

Your answer should be a different plane than the one you found in subpart **(i)**. (This is an important point: since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent, any pair of them span a plane, but all three pairs of them span different planes.)

(iii) The planes you find in subparts **(i)** and **(ii)** intersect at a **line**. Solve for the equation of this line of intersection in parametric form. What do you notice about the equation of the line?

To help you visualize this line of intersection, use the supplemental Jupyter Notebook.

(iv) In standard form, find the equation of the plane spanned by $\vec{v}_2$ and $\vec{v}_3$. Now, find the intersection of this plane with the object from subpart **(iii)**. What type of geometric object is this new intersection?

Once again, use the supplemental Jupyter Notebook to visualize this intersection.

Problem 4: Finding a Linearly Independent Subset (7 pts)

In each of the parts below, using the algorithm mentioned in **this section of Chapter 2.4**, find **a** linearly independent set of vectors that spans the same span as the given set of vectors.

In your solutions, show all of the steps of the algorithm, clearly state what the vectors in the linearly independent set are, and how many vectors are in the set.

There are multiple possible answers for each part, but all of them have the same number of vectors.

a) (3 pts)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

b) (4 pts)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Problem 5: Rows and Columns (12 pts)

Soon, we will start to learn about matrices. In this problem, we'll start to connect what we've learned about vectors and spans to matrices.

In this question, we'll consider the matrix A below.

$$A = \begin{bmatrix} 5 & 3 & 5 & 2 \\ 3 & 0 & -6 & 4 \\ -2 & 0 & 4 & 3 \\ 8 & 2 & -6 & -8 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

A has 5 rows and 4 columns. There are two ways of looking at A:

1. As a collection of **4 "column" vectors**, each in $\mathbb{R}^5$, stacked side-by-side.
2. As a collection of **5 "row" vectors**, each in $\mathbb{R}^4$, stacked on top of each other.

a) (3 pts) Find a linearly independent set of vectors in $\mathbb{R}^5$ with the same span as the column vectors of A. How many vectors are in this set?

b) (4 pts) Find a linearly independent set of vectors in $\mathbb{R}^4$ with the same span as the row vectors of A. How many vectors are in this set?

You should have found that the number of vectors you found in both parts is the same. This is not a coincidence, it is true for any matrix — the number of linearly independent columns is the same as the number of linearly independent rows. This number is called the **rank** of the matrix.

If you were to run the following Python code, the number you'd see back is the number of linearly independent vectors you found in both parts.

```
import numpy as np

A = np.array([[5, 3, 5, 2],
              [3, 0, -6, 4],
              [-2, 0, 4, 3],
              [8, 2, -6, -8],
              [1, 1, 3, 0]])

np.linalg.matrix_rank(A)
```

c) (3 pts) Open the **the supplemental Jupyter Notebook** we've created for Homework 4, which can either be found **here** on DataHub, or **here** in the course GitHub repository.

Complete the three tasks within orange lines, related to the introduction we've provided above. Include screenshots of your code and its output as part of your PDF.

d) (2 pts) Using what you observed in the notebook, **by hand** (that is, without using Python), compute the result of the following matrix-vector multiplication:

$$\begin{bmatrix} 2 & 4 & 5 \\ 3 & 9 & 8 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

Before working on the next two problems, it's a good idea to review the section on linear independence in Chapter 2.4.

Problem 6: Linear Independence of New Vectors (6 pts)

Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^n$ **are linearly independent**. In both parts below, determine if the new set of vectors is linearly independent. If they are, prove that they are by showing that the only solution to the equation

$$a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 = \vec{0}$$

is $a = b = c = 0$. If they are not, show that there exist scalars a, b, c such that $a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 = \vec{0}$ where at least one of a, b, c is nonzero.

a) (3 pts) $\vec{u}_1 = \vec{v}_2 - \vec{v}_3$, $\vec{u}_2 = \vec{v}_1 - \vec{v}_3$, and $\vec{u}_3 = \vec{v}_1 - \vec{v}_2$

b) (3 pts) $\vec{u}_1 = \vec{v}_2 + \vec{v}_3$, $\vec{u}_2 = \vec{v}_1 + \vec{v}_3$, and $\vec{u}_3 = \vec{v}_1 + \vec{v}_2$

Problem 7: Thinking in Higher Dimensions (8 pts)

a) (3 pts) Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_8$ are 8 vectors in $\mathbb{R}^5$. Fill in each blank below with one of the provided options, and explain your reasoning.

(i) These vectors _____ span all of $\mathbb{R}^5$.
(options: do, do not, might)

(ii) These vectors _____ linearly independent.
(options: are, are not, might be)

(iii) Any 5 of these vectors _____ span all of $\mathbb{R}^5$.
(options: do, do not, might)

b) (5 pts) Suppose $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{10}$ are 10 non-zero vectors in $\mathbb{R}^{11}$.

Furthermore, suppose that $\text{span}(\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{10}\})$ is a 6-dimensional subspace of $\mathbb{R}^{11}$. This means that there exists a subset of 6 of these vectors that is linearly independent and spans the same 6-dimensional subspace as the original 10 vectors; we just don't know which 6.

(i) Let k be the dimension of the subspace spanned by a subset of 4 of these vectors. What are all possible values of k?

(ii) Let m be the dimension of the subspace spanned by a subset of 7 of these vectors. What are all possible values of m?