Homework 6: Rank and Inverses

EECS 245, Fall 2025 at the University of Michigan

due Friday, October 17th, 2025 at 11:59PM Ann Arbor Time

> Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Homeworks are due to Gradescope by 11:59PM on the due date. See the **syllabus** for details on the slip day policy.
>
> Homework will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain and justify your conclusions, using sound reasoning. Your goal should be to convince the reader of your assertions. If a question does not require explanation, it will be explicitly stated.
>
> Before proceeding, make sure you're familiar with the **collaboration policy**.

Total Points: 9 + 16 + 12 + 12 + 5 + 5 + 22 = 81

Problem 1: Rank-Nullity Practice (9 pts)

Recall from **Chapter 2.8** that the rank-nullity theorem states that for any $n \times d$ matrix A,

$$\text{rank}(A) + \dim(\text{nullsp}(A)) = d$$

In each part, identify whether the statement is true or false, and explain why.

a) (3 pts) There exists a 4×5 matrix A with $\text{rank}(A) = 4$ and $\dim(\text{colsp}(A)) = 3$.

b) (3 pts) There exists a 4×5 matrix B with $\text{rank}(B) = 3$ and $\dim(\text{nullsp}(B)) = 2$.

c) (3 pts) There exists a 4×5 matrix C with $\dim(\text{nullsp}(C)) = 4$ and $\dim(\text{nullsp}(C^T)) = 1$.

Problem 2: Spaces (16 pts)

a) (4 pts) Find a matrix A such that

$$\text{nullsp}(A) = \text{span}\left(\left\{\begin{bmatrix}2\\1\\0\\1\end{bmatrix}\right\}\right)$$

What is $\text{rank}(A)$?

b) (4 pts) Find a matrix A such that

$$\text{nullsp}(A) = \text{span}\left(\left\{\begin{bmatrix}2\\2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\1\\0\\1\end{bmatrix}\right\}\right)$$

What is $\text{rank}(A)$?

c) (4 pts) Find a matrix A such that

$$\begin{bmatrix}1\\1\\5\end{bmatrix} \in \text{colsp}(A), \quad \begin{bmatrix}0\\3\\1\end{bmatrix} \in \text{colsp}(A), \quad \begin{bmatrix}1\\1\\2\end{bmatrix} \in \text{nullsp}(A)$$

What is $\text{rank}(A)$?

d) (4 pts) Let $\vec{u} = \begin{bmatrix}1\\3\\4\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}8\\-2\\3\end{bmatrix}$. Explain why there **does not** exist a matrix A such that

$$\vec{u} \in \text{colsp}(A), \quad \vec{v} \in \text{nullsp}(A^T)$$

and propose one change we could make to $\vec{v}$ that would allow such an A to exist.

Hint: In Chapter 2.8, one of the exercises discusses a fact related to this question. But, if you want to make a claim about the required relationship between $\vec{u}$ and $\vec{v}$, you need to derive it from scratch here. This shouldn't take too many lines of work.

Problem 3: Numbers of Solutions (12 pts)

Recall that if A is an $n \times d$ matrix, $\vec{x} \in \mathbb{R}^d$, and $\vec{b} \in \mathbb{R}^n$, then

$$A\vec{x} = \vec{b}$$

is a system of n equations in d unknowns, where the unknowns are the components of $\vec{x}$, i.e. x_1, x_2, $\ldots$, x_d. Solving this system is equivalent to writing $\vec{b}$ as a **linear combination of the columns of A**.

In each part, do two things:

1. Construct a matrix A for which the number of solutions (that is, number of valid $\vec{x}$'s) to the system $A\vec{x} = \vec{b}$ is the number provided.
2. Determine whether the function $f(\vec{x}) = A\vec{x}$ is one-to-one, onto, both, or neither. (See **Chapter 2.9** for a refresher on the definitions.)

The first part has been done for you as an example.

a) 0 or 1, depending on $\vec{b}$

Solution:

1. If there are either 0 or 1 solutions, we know that A's columns must be linearly independent. This is because if a given set of vectors is linearly independent, then any linear combination of them can only be written in one way (a fact that we proved in Chapter 2.6). A's columns must also **not** span all of $\mathbb{R}^n$, since there are some $\vec{b} \in \mathbb{R}^n$ with no solutions for $\vec{x}$, so A must have fewer columns than rows.

   One possible A is $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$. For example, $\vec{b} = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$ only has a single solution for $\vec{x}$, which is $\vec{x} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, while $\vec{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has no solution for $\vec{x}$.

2. The function $f(\vec{x}) = A\vec{x}$ is one-to-one, but not onto. It is one-to-one because of the fact that any linear combination of A's columns can only be written in one way, so if $\vec{x} \neq \vec{y}$, then $A\vec{x}$ and $A\vec{y}$ must also be different. It is not onto, since there are vectors in $\mathbb{R}^3$ (like $\vec{c}$ above) that aren't the output of $f(\vec{x})$.

b) (4 pts) ∞, no matter what $\vec{b}$ is

c) (4 pts) 0 or ∞, depending on what $\vec{b}$ is

d) (4 pts) 1, no matter what $\vec{b}$ is

Problem 4: Projecting onto a Single Vector (12 pts)

In **Homework 5, Problem 4**, you found that the 2×2 matrix P that projects $\vec{u} \in \mathbb{R}^2$ onto the unit vector $\vec{v} \in \mathbb{R}^2$ was

$$P = \begin{bmatrix} v_1^2 & v_1 v_2 \\ v_1 v_2 & v_2^2 \end{bmatrix}$$

a) (6 pts) Find:

(i) $\text{rank}(P)$

(ii) A basis for $\text{colsp}(P)$

(iii) A basis for $\text{nullsp}(P)$

(iv) A basis for $\text{colsp}(P^T)$

b) (3 pts) Explain why P is not invertible, and then explain why the transformation $f(\vec{u}) = P\vec{u}$ can't be reversed.

c) (3 pts) As mentioned in Homework 5, Problem 4, P is an idempotent matrix, meaning that

$$P^2 = P$$

In general, if $P^2 = P$ and P is invertible, what must P be?
Hint: Multiply both sides of $P^2 = P$ by P^{-1}.

Problem 5: Invertibility of XX^T (5 pts)

In **Chapter 2.9**, we proved that $\text{rank}(X) = \text{rank}(X^TX)$. Here, we'll ask you to prove something similar involving XX^T. Note that X^TX is a matrix containing the dot products of all pairs of X's **columns**, while XX^T is a matrix containing the dot products of all pairs of X's **rows**. (This has fact had something to do with **Homework 5, Problem 5**!)

Suppose X is an $n \times d$ matrix, and XX^T is invertible. Find and explain **all** inequalities that **must** be true between n, d, and $r = \text{rank}(X)$.

Problem 6: Trickster (5 pts)

Find a matrix A that is **not equal to the identity matrix**, but where $A^6 = I$.

Once you think of your answer, you should explain how you found it, and should use Python to verify that $A^6 = I$ holds. Include a screenshot of your Python code.

*Hint: This problem has a "trick" to it, and to think of it, I'd suggest re-reading **Chapter 2.9** and watching Lecture 13. The solution doesn't involve very much algebra. Think outside of the box!*

Problem 7: Sherman-Morrison Inverse (22 pts)

In EECS 280 or EECS 281, you may have learned about **memoization**, which involves storing results of an earlier calculation to help speed up future calculations. This problem involves something similar.

Suppose we have an $n \times n$ matrix A whose inverse, A^{-1}, we already know. Remember that finding inverses in general is a difficult task, so once we've found one, we'd like to avoid having to invert again.

And, suppose that we need to know the inverse of

$$B = A + \vec{u}\vec{v}^T$$

which is the sum of A and a rank 1 matrix created by taking the "outer product" of $\vec{u}, \vec{v} \in \mathbb{R}^n$ (as discussed in **Chapter 2.8**). Think of B as a small update to A.

The Sherman-Morrison formula states that

$$B^{-1} = (A + \vec{u}\vec{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\vec{u}\vec{v}^T A^{-1}}{1 + \vec{v}^T A^{-1}\vec{u}}$$

The formula allows us to find the inverse of $A + \vec{u}\vec{v}^T$ just by knowing $\vec{u}$, $\vec{v}$, and A^{-1}, meaning we don't need to recompute the inverse!

a) (3 pts) To start, we'll consider a simpler case of the Sherman-Morrison formula, where $A = I$, the identity matrix. Then, since $I = I^{-1}$, the matrix we're inverting is

$$B = A + \vec{u}\vec{v}^T$$

and its inverse is

$$B^{-1} = (I + \vec{u}\vec{v}^T)^{-1} = I - \frac{\vec{u}\vec{v}^T}{1 + \vec{v}^T\vec{u}}$$

B is invertible, **except when** the denominator of the fraction above is 0, i.e. $1 + \vec{v}^T\vec{u} = 0$. When $1 + \vec{v}^T\vec{u} = 0$, what is true about B?

Hint: Evaluate $B\vec{u}$. What does the result tell you about nullsp(B)?

b) (4 pts) Prove that as long as $1 + \vec{v}^T\vec{u} \neq 0$, that

$$(I + \vec{u}\vec{v}^T)\left(I - \frac{\vec{u}\vec{v}^T}{1 + \vec{v}^T\vec{u}}\right) = I$$

(Yes, this involves a fair bit of algebra.)

c) (8 pts) Now, let's return to the full-fledged Sherman-Morrison formula, where

$$B = A + \vec{u}\vec{v}^T, \qquad B^{-1} = (A + \vec{u}\vec{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\vec{u}\vec{v}^T A^{-1}}{1 + \vec{v}^T A^{-1}\vec{u}}$$

Open the **the supplemental Jupyter Notebook** we've created for Homework 6, which can either be found **here** on DataHub, or **here** in the course GitHub repository.

There, you're asked to implement the Sherman-Morrison formula, and run some experiments to quantify how much quicker using the formula is than computing the inverse of B from scratch.

This problem is **not autograded**. Rather, in your submission to this part, include screenshots of your implementations of functions `generate_random_data`, `invert_B_directly`, `invert_B_with_sherman_morrison`, `run_one_experiment`, and `many_experiments_mean_sd`, along with their outputs on the provided examples.

d) (4 pts) Include screenshots of the code you used to call `many_experiments_mean_sd` for the values provided in the question, the outputs of the print statements you were asked to add, and the `plotly` line chart you were asked to create.

e) (3 pts) Answer the question posed at the end of the supplemental Jupyter Notebook. (This shouldn't be a screenshot; just write your answer in this PDF the same way you answered Problems 1-6.)

If you're curious, look into **Low-Rank Adaptions (LoRA)**, a relatively recent development in large language model research! The general idea is the same as we've worked with here.