Lab 3: Vectors and the Dot Product

EECS 245, Fall 2025 at the University of Michigan

due by the end of your lab section on Wednesday, September 10th, 2025

Name: ____________________

uniqname: ____________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

Activity 1: Norms

Let $\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$. Since both of these vectors have two components, we say they are in $\mathbb{R}^2$, or $\vec{u}, \vec{v} \in \mathbb{R}^2$.

Recall from Chapter 2.1 that the **norm** — or **length** — of a vector $\vec{v} \in \mathbb{R}^n$ is given by:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

a) Find $\|\vec{u}\|$.

b) Find $\frac{\vec{u}}{\|\vec{u}\|}$. This is what's called a **unit vector**, i.e. a vector with a length of 1, that points in the same direction as $\vec{u}$.

c) Find $\|-\vec{u} - 3\vec{v}\|$.

d) In tomorrow's lecture (and Chapter 2.2), we will learn about the **triangle inequality**, which states that for any two vectors $\vec{u}$ and $\vec{v}$,

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

For the vectors $\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ **and** $\vec{v} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$ **from the previous page**, verify that the triangle inequality holds. That is, show that the left-hand side is less than or equal to the right-hand side.

e) Find two **different** vectors in $\vec{x}, \vec{y} \in \mathbb{R}^2$ such that the triangle inequality achieves **equality**, i.e. where

$$\|\vec{x} + \vec{y}\| = \|\vec{x}\| + \|\vec{y}\|$$

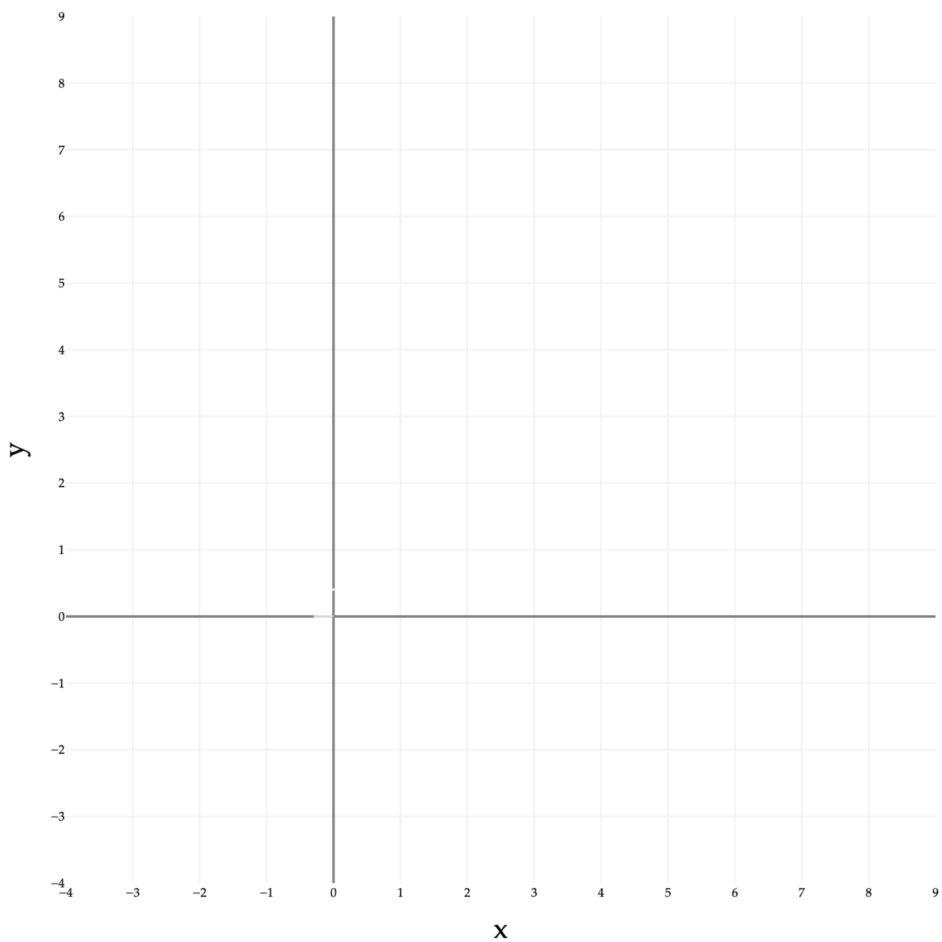
What is the relationship between the $\vec{x}$ and $\vec{y}$ you found?

Activity 2: Linear Combinations

Let $\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$ as in the previous activity.

a) Using the grid below, draw and label the following vectors:

$$\vec{u} \qquad 2\vec{u} \qquad 2\vec{u}+\vec{v} \qquad \vec{v} \qquad \vec{u}+\vec{v} \qquad \vec{u}-\vec{v} \qquad -\vec{u}-3\vec{v}$$



For the rest of this activity, let $\vec{w} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$.

b) Find values of a and b such that $a \begin{bmatrix} 4 \\ 3 \end{bmatrix} + b \begin{bmatrix} -1 \\ -3 \end{bmatrix} = \vec{w}$.

By finding a and b, you have written $\vec{w}$ as a **linear combination** of $\vec{u}$ and $\vec{v}$.

c) Now, try and write $\vec{w}$ as a linear combination of $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. In other words, try and find values of a, b, and c such that

$$a \begin{bmatrix} 4 \\ 3 \end{bmatrix} + b \begin{bmatrix} -1 \\ -3 \end{bmatrix} + c \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{w}$$

What happens? Why?

d) Now, try and write $\vec{w}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$. What happens? Why?

Activity 3: The Dot Product

In tomorrow's lecture — and Chapter 2.2 — we will learn about the **dot product**. The dot product of two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is defined as

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

For example, using $\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$, we have $\vec{u} \cdot \vec{v} = 4 \cdot (-1) + 3 \cdot (-3) = -4 - 9 = -13$.

Note that the dot product of two vectors is a **scalar**, not a vector!

For each pair of vectors below (1) draw them on the grid at the bottom of the page and (2) compute their dot product.

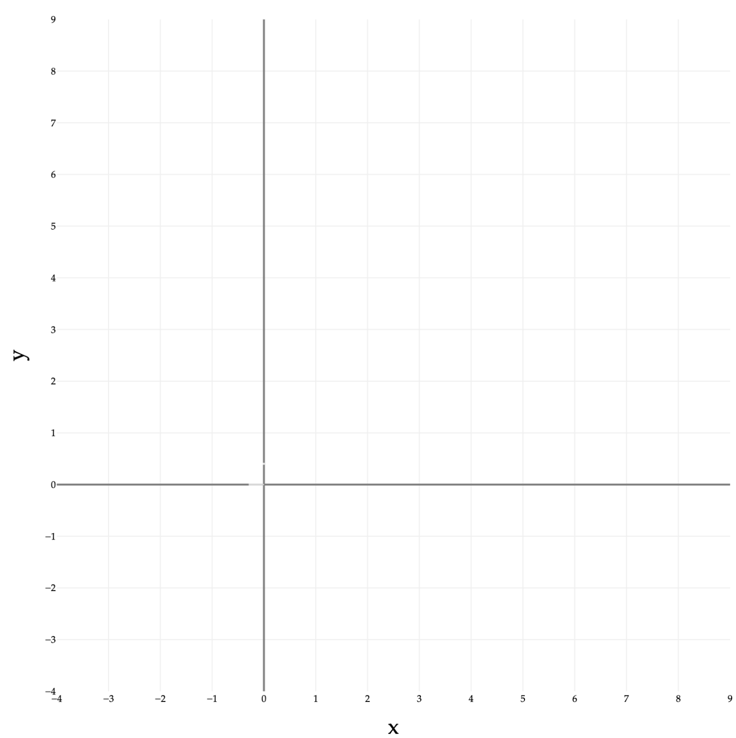
a) $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b) $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 0 \end{bmatrix}$

c) $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$

d) $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$

e) $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$



Activity 4: Angles and Orthogonality

In part **e)** of the previous activity, the vectors $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ had a dot product of 0. When you drew them out, you may have noticed they are at a right angle to each other, or are **orthogonal**.

Formally, we say two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are **orthogonal** if their dot product is 0, i.e. $\vec{u} \cdot \vec{v} = 0$. This definition holds even for vectors in more than 3 dimensions, where we can't visualize the vectors directly.

It may not be clear why angles have anything to do with the dot product. It turns out that the dot product has an equivalent definition:

$$\vec{u} \cdot \vec{v} = \underbrace{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}_{\text{definition from last page}} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$. **Chapter 2.2** has a proof of this equivalence.

Let $\vec{w} = \begin{bmatrix} 5 \\ 0 \\ -4 \\ 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 9 \\ 1 \\ 2 \\ 3 \end{bmatrix}$.

a) Find $\vec{w} \cdot \vec{x}$, $\|\vec{w}\|$, and $\|\vec{x}\|$.

b) Using the results of part **a)**, find the angle between $\vec{w}$ and $\vec{x}$. Leave your answer in the form $\cos^{-1}(\cdot)$.

c) What is $\cos(90°)$? What does this have to do with orthogonality?

Activity 5: Sum–Difference Orthogonality

Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 5 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -3 \end{bmatrix}$.

a) Show that $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are orthogonal.

b) Now suppose $\vec{u}, \vec{v} \in \mathbb{R}^n$ are arbitrary vectors with the same number of components. Is it always true that $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are orthogonal?

- If so, prove why.
- If not, specify conditions under which it's guaranteed that $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are orthogonal.

Hint: Use the distributive property of the dot product, which states that

$$(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$$

Activity 6: Arrays in NumPy

Instead of writing code in a separate Jupyter Notebook for this lab, you will interact with the code cells that exist in the course notes.

In particular, go to **Chapter 2.1** of the course notes, scroll all the way to the bottom, and complete **Activity 9** there. To get checked off, show your lab TA that you've completed the activity — there's no need to submit your code anywhere.