Lab 4: Projections and Spans

EECS 245, Fall 2025 at the University of Michigan

due by the end of your lab section on Wednesday, September 17th, 2025

Name: ____________________

uniqname: ____________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

Activity 1: Presidential Speeches and Cosine Similarity

Complete the tasks in the `lab04.ipynb` notebook, which you can either access through the DataHub link on the course homepage or by pulling our GitHub repository. To receive credit for Activity 1, you'll need to submit your completed `lab04.ipynb` notebook to Gradescope and show your lab TA that all test cases have passed. Instructions on how to do this are in the lab notebook.

Activity 2: Projections

In this activity, we'll explore (orthogonal) projections, as first introduced in Chapter 2.3.

Let $\vec{c} = \begin{bmatrix} 1 \\ 2 \\ -4 \\ 0 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix}$.

a) Find the projection of $\vec{c}$ onto $\vec{d}$. Call this vector $\vec{q}$.

b) Find the error vector, $\vec{r} = \vec{c} - \vec{q}$. Which vector is $\vec{r}$ orthogonal to, $\vec{c}$ or $\vec{d}$? Draw a rough picture of the relationship between $\vec{c}$, $\vec{d}$, $\vec{q}$, and $\vec{r}$.

Activity 3: Orthogonal Decomposition

a) Let $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Write $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, and verify that your answer is correct.

b) In general, suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are **orthogonal** vectors in $\mathbb{R}^n$, meaning that $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$. Given that it is possible to write $\vec{u}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$, show that the coefficients of the linear combination

$$\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_d\vec{v}_d$$

are given by

$$a_i = \frac{\vec{u} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

Hint: Start by taking the dot product of both sides of the linear combination equation with $\vec{v}_1$. What do you notice?

Activity 4: Projections and Norms

a) Suppose $\vec{u}$ and $\vec{v}$ are two **unit vectors** in $\mathbb{R}^n$ — meaning that $\|\vec{u}\| = 1$ and $\|\vec{v}\| = 1$ — and that the angle between them is α.

Show that the projection of $\vec{u}$ onto $\vec{v}$ is $\vec{p} = (\cos\alpha)\,\vec{v}$, and that the projection of $\vec{v}$ onto $\vec{u}$ is $\vec{q} = (\cos\alpha)\,\vec{u}$.

b) Now, suppose that $\vec{u}, \vec{v} \in \mathbb{R}^n$ are arbitrary vectors, not necessarily unit vectors, and suppose $\vec{p}$ is the projection of $\vec{u}$ onto $\vec{v}$.

- Is it possible for $\vec{p}$ to be longer than $\vec{u}$? If so, give an example. If not, prove why not.
- Is it possible for $\vec{p}$ to be longer than $\vec{v}$? If so, give an example. If not, prove why not.

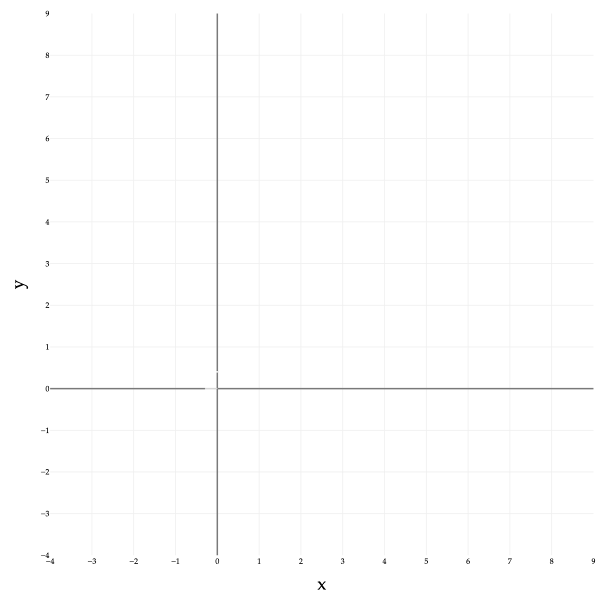
Activity 5: Lines in $\mathbb{R}^n$

In Chapter 2.4 — and yesterday's lecture — we saw that a line in $\mathbb{R}^n$ can be expressed in **parametric** form as

$$L = \vec{p} + t\vec{v}, t \in \mathbb{R}$$

where $\vec{p}$ is a point on the line, and $\vec{v}$ is a vector that points in the direction of the line. t is a free variable; different values for t will give us different points on L.

a) On the grid below, draw the vector $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, the vector $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$, and the line $L = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 6 \\ -3 \end{bmatrix}, t \in \mathbb{R}$.



b) Express the line $L = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 6 \\ -3 \end{bmatrix}, t \in \mathbb{R}$ in the "standard" form for lines in $\mathbb{R}^2$, $y = mx + b$. (Remember that only lines in $\mathbb{R}^2$ can be expressed in this form; in higher dimensions, we need to use the parametric form. Think about why this is the case, and consult Chapter 2.4.)

c) Why is the line $L = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 6 \\ -3 \end{bmatrix}, t \in \mathbb{R}$ **not** equal to the span of any one vector in $\mathbb{R}^2$?

d) Find a line in $\mathbb{R}^4$ that passes through $(0, 1, 2, 3)$ and is **orthogonal** to $\begin{bmatrix} 9 \\ 3 \\ 1 \\ -5 \end{bmatrix}$.

Activity 6: Planes

An important idea from Chapter 2.4 is that two non-parallel vectors in $\mathbb{R}^n$ (where $n \geq 2$) span a plane in n-dimensional space. Here, we'll show you how to find the equation of such a plane, given two vectors in $\mathbb{R}^3$.

a) Given two vectors $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in $\mathbb{R}^3$, show that the vector $\vec{q}$ (defined below) is orthogonal to both $\vec{a}$ and $\vec{b}$.

$$\vec{q} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

The vector $\vec{q}$ is called the **cross product** of $\vec{a}$ and $\vec{b}$. The cross product is only defined for two vectors in $\mathbb{R}^3$ specifically, and the product is another vector in $\mathbb{R}^3$. (This differentiates it from the dot product, which is defined for two vectors in any $\mathbb{R}^n$, and whose output is a scalar.)

b) Find the cross product of $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

c) Let $\vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$ be your answer to part **b)**.

Verify that the points $(0, 0, 0)$, $(2, -1, 3)$ and $(1, 2, -1)$ satisfy the equation

$$q_1 x + q_2 y + q_3 z = 0$$

Once you're done, if you'd like, you could rearrange $q_1 x + q_2 y + q_3 z = 0$ to get something of the form $z = f(x, y)$.

d) Above, we wrote the equation of the plane spanned by $\vec{v}_1$ and $\vec{v}_2$ in the "standard form" for planes in $\mathbb{R}^3$, $ax + by + cz + d = 0$ (where $d = 0$).

Now, write the equation of the plane spanned by $\vec{v}_1$ and $\vec{v}_2$ in **parametric** form. This won't require much work; it's more that we want you to understand that there are two ways of expressing planes in $\mathbb{R}^3$.

e) (Optional) Using just what we've covered in this lab, this final part may be quite challenging at first. But, try it when you get a chance!

In $\mathbb{R}^2$, any two points uniquely determine a line. In $\mathbb{R}^3$, any three points uniquely determine a plane.

Consider the points $(3, 4, 5)$, $(1, 9, -2)$, and $(2, 2, 0)$. Find the equation of the plane that passes through all three points, and express that plane in (1) parametric form and (2) standard form, $ax + by + cz + d = 0$.

Hint: The plane does not necessarily pass through the origin, unlike the plane we found in part ***c)****, which had to pass through the origin by virtue of being the span of a set of vectors.*

The parametric form is much easier to find — start with it, and use your answer to find the equation in standard form.