Lab 5: Vector Spaces, Subspaces, and Bases

EECS 245, Fall 2025 at the University of Michigan

due by the end of your lab section on Wednesday, September 24th, 2025

Name: ______________________________

uniqname: ______________________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

Acknowledgements: Activities 1, 3, and 6 are taken from **here**, and Activity 4 is taken from *Linear Algebra* by Gilbert Strang. Consider looking at these sources for more practice problems.

Activity 1: Linear Independence

Let $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

a) Find scalars a, b, c, and d such that $a\vec{w} + b\vec{x} + c\vec{y} + d\vec{z} = \vec{0}$, and at least one of the scalars is non-zero. By doing so, you're showing that $\vec{w}, \vec{x}, \vec{y}, \vec{z}$ are linearly dependent.

Solution:
The systematic way to do this is to write out the vector equation as a system of equations:

$$a\vec{w} + b\vec{x} + c\vec{y} + d\vec{z} = \vec{0}$$

is equivalent to:

$$a \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which is equivalent to:

$$\begin{aligned} a + b + c + d &= 0 \\ a + d &= 0 \\ b + c &= 0 \\ c + d &= 0 \end{aligned}$$

Equation (2) tells us $a = -d$, equation (4) tells us $c = -d$, and equation (3) tells us $b = -c = -(-d) = d$. So, solutions for the coefficients are of the form $a = -d, b = d, c = -d$, for any $d \in \mathbb{R}$. The simplest choice is to pick $d = 1$, which gives us $a = -1, b = 1, c = -1$, and indeed we can verify that

$$-\vec{w} + \vec{x} - \vec{y} + \vec{z} = \begin{bmatrix} -1+1 \\ -1+1 \\ 1-1 \\ -1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$$

b) Find scalars A, B, and C such that $\vec{z} = A\vec{w} + B\vec{x} + C\vec{y}$. This is another way of showing that $\vec{w}, \vec{x}, \vec{y}, \vec{z}$ are linearly dependent.

Solution:
Using the fact that $-\vec{w}+\vec{x}-\vec{y}+\vec{z}=\vec{0}$, we can write

$$\vec{z}=\vec{w}-\vec{x}+\vec{y}$$

So, $A=1, B=-1, C=1$. This is just one of the many ways to write any one of these vectors as a linear combination of the other three.

c) Show that $\text{span}(\{\vec{w},\vec{x},\vec{y},\vec{z}\}) \neq \mathbb{R}^4$ by finding a vector $\vec{v} \in \mathbb{R}^4$ such that $\vec{v} \notin \text{span}\{\vec{w},\vec{x},\vec{y},\vec{z}\}$.

Solution: Recall, $\text{span}(\{\vec{w},\vec{x},\vec{y},\vec{z}\})$ is the set of all linear combinations of $\vec{w},\vec{x},\vec{y},\vec{z}$. So, any vector in $\text{span}(\{\vec{w},\vec{x},\vec{y},\vec{z}\})$ can be written in the form

$$a\vec{w}+b\vec{x}+c\vec{y}+d\vec{z}=a\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+b\begin{bmatrix}1\\0\\1\\0\end{bmatrix}+c\begin{bmatrix}0\\0\\1\\1\end{bmatrix}+d\begin{bmatrix}0\\1\\0\\1\end{bmatrix}=\begin{bmatrix}a+b\\a+d\\c+d\\b+c\end{bmatrix}$$

All we need to do in this part is find a vector $\vec{v} \in \mathbb{R}^4$ that can't be written in this way. Suppose we choose $a=1, b=2, c=3, d=4$. Then, $a+b=3$, $a+d=5$, $c+d=7$, and $b+c=5$. If we construct the vector $\vec{v}=\begin{bmatrix}3\\5\\7\\5\end{bmatrix}$, it's in $\text{span}(\{\vec{w},\vec{x},\vec{y},\vec{z}\})$ since it's just $\vec{w}+2\vec{x}+3\vec{y}+4\vec{z}$. But if we change one of these components, say the last component from 5 to 4, then we'd need to solve the system

$$\begin{aligned}a+b&=3\\a+d&=5\\c+d&=7\\b+c&=4\end{aligned}$$

but this system will be inconsistent, since the first three equations will satisfy $a=1, b=2, c=3, d=4$, but the last equation will be $b+c=4\neq 5$. So, $\boxed{\vec{v}=\begin{bmatrix}3\\5\\7\\4\end{bmatrix}}$ is in $\mathbb{R}^4$ but not in $\text{span}(\{\vec{w},\vec{x},\vec{y},\vec{z}\})$, which means that $\text{span}(\{\vec{w},\vec{x},\vec{y},\vec{z}\}) \neq \mathbb{R}^4$.

d) Why is the fact that $\text{span}(\{\vec{w},\vec{x},\vec{y},\vec{z}\}) \neq \mathbb{R}^4$ enough to conclude that $\vec{w},\vec{x},\vec{y},\vec{z}$ are linearly **dependent**?

Solution: Any four linearly independent vectors in $\mathbb{R}^4$ must span all of $\mathbb{R}^4$. So, if $\text{span}(\{\vec{w}, \vec{x}, \vec{y}, \vec{z}\}) \neq \mathbb{R}^4$, then since we're dealing with 4 vectors, they must be linearly dependent (since if they linearly independent, they would indeed span all of $\mathbb{R}^4$).

Activity 2: Formal Definition of Linear Independence

Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d \in \mathbb{R}^n$, and that $\vec{b} \in \text{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\})$.

a) Give a one sentence English explanation of what it means for $\vec{b} \in \text{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\})$.

> **Solution:**
> If $\vec{b} \in \text{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\})$, then there exist scalars $a_1, a_2, \ldots, a_d$ such that $\vec{b} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d$, i.e. $\vec{b}$ can be written as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$.

b) Suppose that $a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d = \vec{b}$ **and** $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_d\vec{v}_d = \vec{b}$, where at least one of the a_i's is different from its corresponding c_i.
Using the formal definition of linear independence from Chapter 2.4, determine whether or not $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are linearly independent, and prove your answer.

> **Solution:**
> We're given that
>
> $$a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d = \vec{b}$$
> $$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_d\vec{v}_d = \vec{b}$$
>
> Subtracting the two equations gives us
>
> $$(a_1 - c_1)\vec{v}_1 + (a_2 - c_2)\vec{v}_2 + \ldots + (a_d - c_d)\vec{v}_d = \vec{0}$$
>
> We know that vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are linearly independent if the only way to write the zero vector $\vec{0}$ as a linear combination of them is to have all the coefficients be zero.
>
> But here, we were told that at least one of the a_i's is different from its corresponding c_i, meaning that at least one of the $(a_i - c_i)$ values is non-zero. This means that there is some way to create $\vec{0}$ using a non-zero linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$, which means that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are linearly dependent.

c) Find another set of coefficients $k_1, k_2, \ldots, k_d$ such that

$$k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots + k_d\vec{v}_d = \vec{b}$$

and at least one of the k_i's is different from its corresponding a_i or c_i.

By doing this, you're showing that if there is at least one way to write $\vec{b}$ as a linear combination of a set of vectors, then there are infinitely many ways to write $\vec{b}$ as a linear combination of those vectors; there can't just be two or three ways to do it.

Solution:
In the previous proof we subtracted the following two equations. What if we add them?

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d = \vec{b}$$
$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_d\vec{v}_d = \vec{b}$$

This would give us

$$(a_1 + c_1)\vec{v}_1 + (a_2 + c_2)\vec{v}_2 + \ldots + (a_d + c_d)\vec{v}_d = 2\vec{b}$$

Dividing both sides by 2 gives us

$$\left(\frac{a_1 + c_1}{2}\right)\vec{v}_1 + \left(\frac{a_2 + c_2}{2}\right)\vec{v}_2 + \ldots + \left(\frac{a_d + c_d}{2}\right)\vec{v}_d = \vec{b}$$

This is another linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ that equals $\vec{b}$! So $k_1 = \frac{a_1+c_1}{2}, k_2 = \frac{a_2+c_2}{2}, \ldots, k_d = \frac{a_d+c_d}{2}$.

Why does this imply that there are infinitely many ways to write $\vec{b}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$? It's because we could repeat this process once again, to get $\frac{a_1+k_1}{2}$, $\frac{a_2+k_2}{2}$, $\ldots$, $\frac{a_d+k_d}{2}$ as coefficients, and then again, and again. There are other ways to write $\vec{b}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ since they're linearly dependent, but we'd need to know more about the specific relationships between the vectors to find more.

Activity 3: Introduction to Subspaces

As discussed in **Chapter 2.6**, a **subspace** S of a vector space V is a subset of V that itself is a vector space, contains the zero vector, and is **closed** under addition and scalar multiplication. That is, if you take any two vectors in in S, any of their linear combinations must also be in S.

Only one of the following is a subspace of $\mathbb{R}^3$. Which one? Explain why the others are not subspaces.

The set of vectors $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ such that

(i) $x + 2y - 3z = 4$

(ii) $\vec{v}$ is on the line $L = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, t \in \mathbb{R}$

(iii) $x + y + z = 0$ and $x - y + z = 1$

(iv) $x = -z$ and $x = z$

(v) $x^2 + y^2 = z$

Solution:
Recall that a subspace must contain the zero vector and must be closed under addition and scalar multiplication.

(i) $x + 2y - 3z = 4$ is **not a subspace**. The zero vector is not in the set, since plugging in $x = 0, y = 0, z = 0$ to the equation $x + 2y - 3z = 4$ gives us $0 + 0 - 0 = 4$, which is not true. $x + 2y - 3z = 4$ is a plane in $\mathbb{R}^3$, and planes are subspaces only when they contain the zero vector.

(ii) The line $L = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, t \in \mathbb{R}$ is **not a subspace**. The zero vector is not in the set, since no value of t makes $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The first equation implies $1 + 2t = 0 \implies t = -\frac{1}{2}$, while the last implies $0 + 4t = 0 \implies t = 0$, which is a contradiction.

(iii) $x + y + z = 0$ and $x - y + z = 1$ is **not a subspace**. These are two non-parallel planes in $\mathbb{R}^3$, which means their intersection is a line in $\mathbb{R}^3$. Lines are subspaces only when they pass through the origin, i.e. contain the zero vector. But the second equation requires $x - y + z = 1$, but at $(0, 0, 0)$ this is $0 - 0 + 0 = 1$, which is not true, meaning that the zero vector is not in the set and so the set is not a subspace.

(iv) $x = -z$ and $x = z$ $\boxed{\textbf{is a subspace}}$. For $x = -z$ and $x = z$ to both be true, we'd need $z = -z$, which implies $z = 0$ and $x = 0$. So, this is the set of all vectors whose first and third components are 0. The zero vector is in the set (since the zero vector's first and third components are 0), and the set is closed under addition and scalar multiplication, since if

$$\vec{u} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$$

then

$$c\vec{u} + d\vec{v} = \begin{bmatrix} 0 \\ ca + db \\ 0 \end{bmatrix}$$

is also in the set. So, the set of vectors in $\mathbb{R}^3$ that satisfy $x = -z$ and $x = z$ is a subspace.

(v) $x^2 + y^2 = z$ is **not a subspace**. The zero vector is the set, since plugging in $(x, y, z) = (0, 0, 0)$ gives us $0^2 + 0^2 = 0$, which is fine. But, the set is not closed under scalar multiplication. For example, consider $\begin{bmatrix} 3 \\ 4 \\ 25 \end{bmatrix}$, which is in the set, but $2 \begin{bmatrix} 3 \\ 4 \\ 25 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 50 \end{bmatrix}$ is not in the set, since $6^2 + 8^2 = 100 \neq 50$.

Activity 4: Finding Non-Examples of Subspaces

In this activity, you'll find sets of vectors in $\mathbb{R}^2$ that satisfy some, but not all, of the requirements for a subspace. Think creatively, and since we're working in $\mathbb{R}^2$, visualize the vectors!

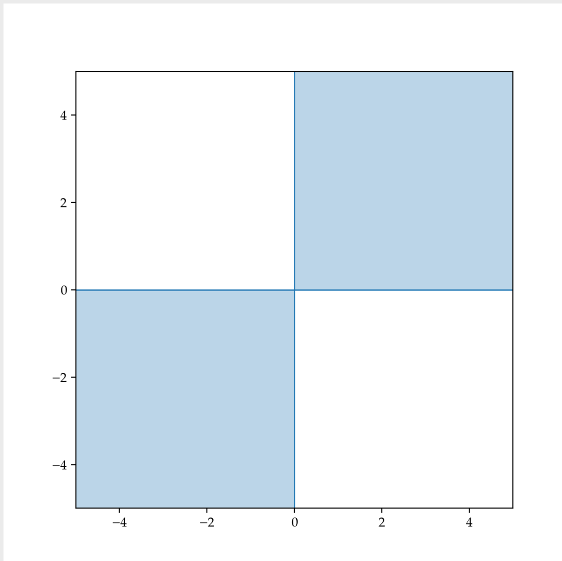
a) Find a set of vectors in $\mathbb{R}^2$ such that the sum of any two vectors $\vec{u}$ and $\vec{v}$ in the set is also in the set, but $\frac{1}{2}\vec{v}$ is possibly not in the set.

> **Solution:**
> One possible answer is the set of all vectors with integer components, e.g.
>
> $$S = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$$
>
> The sum of any two vectors in S is also in S, since the sum of two integers is another integer. However, $\frac{1}{2}\vec{v}$ is not necessarily in S; for example, $\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ is not in S.
> So, this S is a sub**set**, but not a sub**space**.

b) Find a set of vectors in $\mathbb{R}^2$ such that $c\vec{v}$ is in the set for any vector $\vec{v}$ in the set and any scalar c, but the sum of any two vectors $\vec{u}$ and $\vec{v}$ in the set is possibly not in the set.

Solution:
One possible answer is the set of all vectors in which either both components are positive, both components are negative, or both components are zero. In other words, this is the set of all vectors that exist in the top-right and bottom-left quadrants of the xy-plane.

$$S = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in \mathbb{R}, a \geq 0, b \geq 0 \text{ or } a \leq 0, b \leq 0 \text{ or } a = 0, b = 0 \right\}$$



Two vectors in S, for example, are $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (top right) and $\begin{bmatrix} -4 \\ -1 \end{bmatrix}$ (bottom left). Any scalar multiple of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is also in S; $k \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2k \\ 3k \end{bmatrix}$ is in the top-right quadrant if $k > 0$ and in the bottom-left quadrant if $k < 0$.

But, the sum $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ is not in S, since it is in the second quadrant.

Activity 5: Bases

Recall from Chapter 2.6 that a **basis** for a subspace S is a set of vectors that

1. span all of S, **and**
2. are linearly independent

In each part below, find **two different possible bases** for the given vector space, and state the **dimension** of the vector space. (Note that this is effectively what you're doing in **Problems 4 and 5 of Homework 4**, we just hadn't introduced the term "basis" at that point.)

a) $S = \text{span}\left(\left\{\begin{bmatrix}1\\3\\3\end{bmatrix}, \begin{bmatrix}-3\\-9\\-9\end{bmatrix}, \begin{bmatrix}1\\5\\-1\end{bmatrix}, \begin{bmatrix}2\\7\\4\end{bmatrix}, \begin{bmatrix}1\\4\\1\end{bmatrix}\right\}\right)$

Solution: Here, we'll employ the algorithm mentioned at the end of Chapter 2.4 to find a linearly independent subset of S that spans it.
Let's call the set of vectors in our basis B.

- We'll start with $B = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \right\}$.
- $\begin{bmatrix} -3 \\ -9 \\ -9 \end{bmatrix}$ is just $-3 \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$, so we won't add it.
- $\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$ is not a scalar multiple of $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$. We know this because if it were the case that $\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} = k \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$ for some scalar k, then we'd need $1 = k$, $5 = 3k$, and $-1 = 3k$, which are inconsistent. So, we'll add $\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$ to B, which now is $B = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} \right\}$.
- Is $\begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$? To determine whether it is, we'll look for scalars a and b such that

  $$a \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

  This is equivalent to the system

  $$\begin{aligned} a + b &= 2 \\ 3a + 5b &= 7 \\ 3a - b &= 4 \end{aligned}$$

  Subtracting equations 2 and 3 gives $6b = 3 \implies b = \frac{1}{2}$, and plugging this into equation 1 gives $a + \frac{1}{2} = 2 \implies a = \frac{3}{2}$. Let's check if this system is consistent. Evaluating $\frac{3}{2} \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$ gives us

$$\frac{3}{2}\begin{bmatrix}1\\3\\3\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\5\\-1\end{bmatrix}=\begin{bmatrix}3/2\\9/2\\9/2\end{bmatrix}+\begin{bmatrix}1/2\\5/2\\-1/2\end{bmatrix}$$
$$=\begin{bmatrix}2\\7\\4\end{bmatrix}$$

So, $\begin{bmatrix}2\\7\\4\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\3\\3\end{bmatrix}$ and $\begin{bmatrix}1\\5\\-1\end{bmatrix}$, so we won't add it to B. (Remember, the point of B is that it is linearly independent and spans S.)

- What's left is $\begin{bmatrix}1\\4\\1\end{bmatrix}$. Is it a linear combination of $\begin{bmatrix}1\\3\\3\end{bmatrix}$ and $\begin{bmatrix}1\\5\\-1\end{bmatrix}$? To determine whether it is, we'll look for scalars a and b such that

$$a\begin{bmatrix}1\\3\\3\end{bmatrix}+b\begin{bmatrix}1\\5\\-1\end{bmatrix}=\begin{bmatrix}1\\4\\1\end{bmatrix}$$

This is equivalent to the system

$$\begin{aligned}a+b&=1\\3a+5b&=4\\3a-b&=1\end{aligned}$$

Subtracting equations 2 and 3 gives $6b=3 \implies b=\frac{1}{2}$, and plugging this into equation 1 gives $a+\frac{1}{2}=1 \implies a=\frac{1}{2}$. Let's check if this system is consistent. Evaluating $\frac{1}{2}\begin{bmatrix}1\\3\\3\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\5\\-1\end{bmatrix}$ gives us $\begin{bmatrix}1\\4\\1\end{bmatrix}$.

So, $\begin{bmatrix}1\\4\\1\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\3\\3\end{bmatrix}$ and $\begin{bmatrix}1\\5\\-1\end{bmatrix}$, so we won't add it to B.

So, $B=\left\{\begin{bmatrix}1\\3\\3\end{bmatrix},\begin{bmatrix}1\\5\\-1\end{bmatrix}\right\}$ is a linearly independent subset of S that spans S, i.e. it is a basis for S. **The dimension of S is 2.**

If we want another basis, we could just swap out $\begin{bmatrix}1\\3\\3\end{bmatrix}$ for $\begin{bmatrix}1\\4\\1\end{bmatrix}$, the most recent vector we considered adding to B. We didn't add $\begin{bmatrix}1\\4\\1\end{bmatrix}$ to B since it's a linear combination of

$\begin{bmatrix}1\\3\\3\end{bmatrix}$ and $\begin{bmatrix}1\\5\\-1\end{bmatrix}$, but that also means that $\begin{bmatrix}1\\3\\3\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\4\\1\end{bmatrix}$ and $\begin{bmatrix}1\\5\\-1\end{bmatrix}$, meaning that we can create with $\begin{bmatrix}1\\4\\1\end{bmatrix}$ and $\begin{bmatrix}1\\5\\-1\end{bmatrix}$ anything we could create with $\begin{bmatrix}1\\3\\3\end{bmatrix}$ and $\begin{bmatrix}1\\5\\-1\end{bmatrix}$. So, another basis for S is $\left\{\begin{bmatrix}1\\4\\1\end{bmatrix}, \begin{bmatrix}1\\5\\-1\end{bmatrix}\right\}$.

b) $S = \left\{\begin{bmatrix}v_1\\v_2\end{bmatrix} \mid v_1 = -v_2; v_1, v_2 \in \mathbb{R}\right\}$

Solution:
One basis for S is $\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$, since any vector in S is a scalar multiple of $\begin{bmatrix}1\\-1\end{bmatrix}$. The dimension of S is 1.

Another basis for S is $\left\{\begin{bmatrix}-5\\5\end{bmatrix}\right\}$. There's nothing special about the number 5 – replace it with any other non-zero number and you'll get another basis for S.

c) $S = \left\{\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} \mid v_4 = 0; v_1, v_2, v_3 \in \mathbb{R}\right\}$

Solution:

One basis for S is $\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \right\}$, since any vector in S is a linear combination of these three vectors. The dimension of S is 3.

The example basis above is perhaps the simplest possible basis for S, but there are infinitely many other bases for S. For example, other ones are

$$\left\{ \begin{bmatrix}2\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-394\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\15\\0\end{bmatrix} \right\}$$

and

$$\left\{ \begin{bmatrix}3\\5\\2\\0\end{bmatrix}, \begin{bmatrix}0\\-394\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\15\\0\end{bmatrix} \right\}$$

Activity 6: Intersections of Subspaces

Let:

- M be the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix}1\\1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\-4\\1\\5\end{bmatrix}$
- N be the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix}0\\-2\\1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\-1\\1\\3\end{bmatrix}$

a) Find a vector that belongs to both M and N. (In other words, find a vector $\vec{v}$ such that $\vec{v} \in M$ and $\vec{v} \in N$.)

There are infinitely many answers; pick the answer with a first component of 1.

Solution:

$\begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix}$ is a vector in both M and N; it's the sum of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -4 \\ 1 \\ 5 \end{bmatrix}$, and it's also the sum of $\begin{bmatrix} 0 \\ -2 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}$.

Let's suppose you didn't initially notice the fact above. How would we approach the problem more systematically?

Any vector in M is of the form $a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ -4 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ a-4b \\ a+b \\ 5b \end{bmatrix}$. Any vector in N is of the form $c \begin{bmatrix} 0 \\ -2 \\ 1 \\ 2 \end{bmatrix} + d \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} d \\ -2c-d \\ c+d \\ 2c+3d \end{bmatrix}$.

To find a vector that belongs to both M and N, we need to find scalars a, b, c, d such that

$$\begin{aligned} a &= d \boxed{= 1} \\ a - 4b &= -2c - d \\ a + b &= c + d \\ 5b &= 2c + 3d \end{aligned}$$

The boxed $= 1$ comes from the fact that we were told to pick the answer with a first component of 1. So, for this solution, $a = d = 1$.

Plugging these into the third equation gives $1 + b = c + 1$, which implies $b = c$. Plugging this into the second equation gives $1 - 4b = -2b - 1$, which implies $2b = 2$ or $b = 1$, and from the fact that $b = c$ we have $c = 1$.

So, $a = b = c = d = 1$ should give us a vector that belongs to both M and N. Let's check:

$$\underbrace{1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix}}_{\text{shows that this vector is in } M}$$

and

$$\underbrace{1\begin{bmatrix}0\\-2\\1\\2\end{bmatrix}+1\begin{bmatrix}1\\-1\\1\\3\end{bmatrix}=\begin{bmatrix}1\\-3\\2\\5\end{bmatrix}}_{\text{shows that this vector is in } N}$$

So, $\begin{bmatrix}1\\-3\\2\\5\end{bmatrix}$ is in both M and N.

b) Fill in the blanks: the set of all vectors that belong to both M and N is a subspace of $\mathbb{R}^4$ with dimension __________.

Use the space below for scratch work.

Solution:
The blank should be $\boxed{1}$, The set of all vectors that belong to both M and N form a line, or 1-dimensional subspace of $\mathbb{R}^4$.

That line is

$$L = t\begin{bmatrix}1\\-3\\2\\5\end{bmatrix}, \qquad t \in \mathbb{R}$$

Where did this come from? In the previous part, we fixed that the first component of the vector we were looking for was 1. But, if we change that to 2, then we'd have found the solution $a = b = c = d = 2$, which would have told us $\begin{bmatrix}2\\-6\\4\\10\end{bmatrix}$ is in both M and N.

So, any vector in both M and N is a scalar multiple of $\begin{bmatrix}1\\-3\\2\\5\end{bmatrix}$.