Lab 5: Projections and Spans Solutions

EECS 245, Winter 2026 at the University of Michigan

due by the end of your lab section

Name: ______________________________

uniqname: ______________________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

Recap: Projections and Spans

- (Chapter 3.4) The **orthogonal projection** of the vector $\vec{u}$ onto the vector $\vec{v}$ is given by

  $$\vec{p} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$

  Above, the scalar $k^* = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$ was chosen to minimize $\|\vec{u} - k\vec{v}\|^2$.

- The vector $\vec{p}$ is called the orthogonal projection because the resulting error vector,

  $$\vec{e} = \vec{u} - \vec{p} = \vec{u} - k^* \vec{v}$$

  is orthogonal to $\vec{v}$.

  $$\vec{e} \cdot \vec{v} = 0$$

- (4.1) The **span** of a set of vectors is the set of all possible linear combinations of the vectors in the set.

  $$\text{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\}) = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_d\vec{v}_d \mid a_1, a_2, \ldots, a_d \in \mathbb{R}\}$$

  - The span of one vector in $\mathbb{R}^n$ is a line through the origin.
  - The span of two **non-parallel** vectors in $\mathbb{R}^n$ is a plane through the origin; this plane is called a **2-dimensional subspace** of $\mathbb{R}^n$.
  - In general, the span of d vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ of dimension 0 to d, depending on the vectors and their relationships.
  - Think of a d-dimensional subspace of $\mathbb{R}^n$ as a "slice" of $\mathbb{R}^n$ that goes through the origin, in which you can move in d directions.

Activity 1: Presidential Speeches and Cosine Similarity

Complete the tasks in the `lab05.ipynb` notebook, which you can either access through the DataHub link on the course homepage or by pulling our GitHub repository. To receive credit for Activity 1, you'll need to show your lab TA that all test cases have passed. Instructions on how to do this are in the lab notebook.

Activity 2: Orthogonal Projections

Let $\vec{c} = \begin{bmatrix} 1 \\ 2 \\ -4 \\ 0 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix}$. Note that $\|\vec{c}\|^2 = 21$, $\|\vec{d}\|^2 = 14$, and $\vec{c} \cdot \vec{d} = 7$.

a) Find the orthogonal projection of $\vec{c}$ onto $\vec{d}$. Call this vector $\vec{q}$.

Solution:

$$\begin{aligned} \vec{q} &= \left(\frac{\vec{c} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \right) \vec{d} \\ &= \frac{1 \cdot 3 + 2 \cdot 2 + (-4) \cdot 0 + 0 \cdot (-1)}{3^2 + 2^2 + 0^2 + (-1)^2} \vec{d} \\ &= \frac{3+4}{9+4+1} \vec{d} \\ &= \frac{7}{14} \vec{d} \\ &= \frac{1}{2} \vec{d} \\ &= \begin{bmatrix} 1.5 \\ 1 \\ 0 \\ -0.5 \end{bmatrix} \end{aligned}$$

b) Find the error vector, $\vec{r} = \vec{c} - \vec{q}$. Which vector is $\vec{r}$ orthogonal to, $\vec{c}$ or $\vec{d}$? Draw a rough picture of the relationship between $\vec{c}$, $\vec{d}$, $\vec{q}$, and $\vec{r}$.

Solution: First, let's find $\vec{r}$.

$$\begin{aligned}\vec{r} &= \vec{c} - \vec{q} \\ &= \begin{bmatrix} 1 \\ 2 \\ -4 \\ 0 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 1 \\ 0 \\ -0.5 \end{bmatrix} \\ &= \begin{bmatrix} -0.5 \\ 1 \\ -4 \\ 0.5 \end{bmatrix}\end{aligned}$$

$\vec{r}$ is orthogonal to $\vec{d}$, not $\vec{c}$, as confirmed by the dot products. The key idea we introduced in **Chapter 3.4** is that the error vector is orthogonal to the vector we projected onto. Here, $\vec{r}$ is the error vector and $\vec{d}$ is the vector we projected onto.

$$\begin{aligned}\vec{r} \cdot \vec{c} &= (-0.5) \cdot 1 + 1 \cdot 2 + (-4) \cdot (-4) + 0.5 \cdot 0 \\ &= -0.5 + 2 + 16 \\ &= 17.5\end{aligned}$$

$$\begin{aligned}\vec{r} \cdot \vec{d} &= (-0.5) \cdot 3 + 1 \cdot 2 + (-4) \cdot 0 + 0.5 \cdot (-1) \\ &= -1.5 + 2 - 0.5 \\ &= 0\end{aligned}$$

Activity 3: Orthogonal Decomposition

a) Let $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Write $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, and verify that your answer is correct. Note that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are pairwise orthogonal.

Solution:
We are given

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \quad \vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We're looking for scalars a, b, c such that $\vec{u} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$.

Solution 1: Solving a system of equations

$$a \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + b \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + c \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Is equivalent to the system of equations:

$$-a + 2b + 2c = 1 \tag{1}$$
$$2a + 2b - c = 1 \tag{2}$$
$$2a - b + 2c = 1 \tag{3}$$

Using elimination:

$$\begin{aligned} \text{(Eq. 2)} - \text{(Eq. 3)}: \quad & (2a - 2a) + (2b - (-b)) + (-c - 2c) = 0 \\ & \Rightarrow \quad 3b - 3c = 0 \quad \Rightarrow \quad b = c \\ \text{(Eq. 2)} - \text{(Eq. 1)}: \quad & (2a - (-a)) + (2b - 2b) + (-c - 2c) = 0 \\ & \Rightarrow \quad 3a - 3c = 0 \quad \Rightarrow \quad a = c \end{aligned}$$

This tells us that $a = b = c$. Plugging this back into Eq. 2 gives us:

$$-a + 2a + 2a = 1 \rightarrow 3a = 1 \rightarrow a = \frac{1}{3}$$

So, $a = b = c = \frac{1}{3}$, and:

$$\vec{u} = \tfrac{1}{3}\,\vec{v}_1 + \tfrac{1}{3}\,\vec{v}_2 + \tfrac{1}{3}\,\vec{v}_3$$

We can verify that we did this correctly by computing the right-hand side above:

$$\frac{1}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_2 + \frac{1}{3}\vec{v}_3 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/3 + 2/3 + 2/3 \\ 2/3 + 2/3 - 1/3 \\ 2/3 - 1/3 + 2/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{u}$$

Solution 2: Using the fact that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthogonal
We can use the fact that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthogonal to find coefficients a, b, and c by projecting $\vec{u}$ onto each of the $\vec{v}_i$s. This is similar to what was done in the **Orthogonal Decomposition** section of Chapter 3.4.

Let $\vec{p}_i$ be the projection of $\vec{u}$ onto $\vec{v}_i$, for $i = 1, 2, 3$. Then, we have:

$$\vec{p}_1 = \frac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \frac{1 \cdot (-1) + 1 \cdot 2 + 1 \cdot 2}{(-1)^2 + 2^2 + 2^2}\vec{v}_1 = \frac{3}{9}\vec{v}_1 = \frac{1}{3}\vec{v}_1$$

$$\vec{p}_2 = \frac{\vec{u} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = \frac{1 \cdot 2 + 1 \cdot 2 + 1 \cdot (-1)}{2^2 + 2^2 + (-1)^2}\vec{v}_2 = \frac{3}{9}\vec{v}_2 = \frac{1}{3}\vec{v}_2$$

$$\vec{p}_3 = \frac{\vec{u} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3}\vec{v}_3 = \frac{1 \cdot 2 + 1 \cdot (-1) + 1 \cdot 2}{2^2 + (-1)^2 + 2^2}\vec{v}_3 = \frac{3}{9}\vec{v}_3 = \frac{1}{3}\vec{v}_3$$

Adding $\vec{p}_1, \vec{p}_2, \vec{p}_3$ gives us:

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = \frac{1}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_2 + \frac{1}{3}\vec{v}_3 = \vec{u}$$

b) In general, suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are **orthogonal** vectors in $\mathbb{R}^n$, meaning that $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$. **Given that** it is possible to write $\vec{u}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$, show that the coefficients of the linear combination

$$\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_d\vec{v}_d$$

are given by

$$a_i = \frac{\vec{u} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

Hint: Start by taking the dot product of both sides of the linear combination equation with $\vec{v}_i$. What do you notice?

Solution:
We're told to assume that any pair of vectors among $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are orthogonal, and that $\vec{u}$ can be written as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$.

$$\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_d\vec{v}_d$$

As the hint suggests, let's take the dot product of both sides with $\vec{v}_i$, where i is some value in $\{1, 2, \ldots, d\}$.

$$\vec{u} \cdot \vec{v}_i = \big(a_1\vec{v}_1 + \cdots + a_d\vec{v}_d\big) \cdot \vec{v}_i$$

Since $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$, only the $i = j$ term survives:

$$\begin{aligned}
\vec{u} \cdot \vec{v}_i &= \big(a_1\vec{v}_1 + \cdots + a_d\vec{v}_d\big) \cdot \vec{v}_i \\
&= a_1(\vec{v}_1 \cdot \vec{v}_i) + \cdots + a_{i-1}(\vec{v}_{i-1} \cdot \vec{v}_i) + a_i(\vec{v}_i \cdot \vec{v}_i) + a_{i+1}(\vec{v}_{i+1} \cdot \vec{v}_i) + \cdots + a_d(\vec{v}_d \cdot \vec{v}_i) \\
&= a_1(0) + \cdots + a_{i-1}(0) + a_i(\vec{v}_i \cdot \vec{v}_i) + a_{i+1}(0) + \cdots + a_d(0) \\
&= a_i(\vec{v}_i \cdot \vec{v}_i)
\end{aligned}$$

Solving for a_i above gives use

$$\vec{u} \cdot \vec{v}_i = a_i(\vec{v}_i \cdot \vec{v}_i) \implies a_i = \frac{\vec{u} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

Since i was arbitrary, the same calculation holds for any value of i in $\{1, 2, \ldots, d\}$.
What we proved here in part **b)** is that when writing a vector $\vec{u}$ as a linear combination of orthogonal vectors, the coefficients of the linear combination can be found by projecting the vector $\vec{u}$ onto each of the orthogonal vectors and adding the results, rather than solving a system of equations.

Activity 4: Finding a Linearly Independent Subset

(This problem also appears as the final, optional problem in Homework 4. It is meant to prepare you for Problem 4 on Homework 4, which asks a very similar question.)

Recall from Chapter 4.2 that a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ is **linearly independent** if either of the following equivalent conditions hold:

- None of the vectors can be written as a linear combination of the others.
- The only way to create the zero vector as a linear combination of the vectors is if all the coefficients are zero. In other words, the only solution to

  $$a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d = \vec{0}$$

  is $a_1 = a_2 = \ldots = a_d = 0$.

Chapter 4.2 introduces an algorithm for finding a linearly independent subset of a given set of vectors with the same span as the original set:

```
given v_1, v_2, ..., v_d
initialize linearly independent set S = {v_1}
for i = 2 to d:
    if v_i is not a linear combination of S:
        add v_i to S
```

In each of the parts below, find **a linearly independent** set of vectors that spans the same span as the given set of vectors. There are multiple possible answers for each part, but all of them have the same number of vectors.

a)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Solution: $i = 2, S = \{\vec{v_1}\} : \vec{v_2} \notin \text{span}(S)$. There's no way to make $\vec{v_2}$ as a linear combination of v_1, because $\vec{v_1}$ has a 0 in the second component, while $\vec{v_2}$ has a 1. So, we add $\vec{v_2}$ to the set.

$i = 3, S = \{\vec{v_1}, \vec{v_2}\} : \vec{v_3} \notin \text{span}(S)$. Similar idea here, but with the third component instead of the second. Add $\vec{v_3}$ to the set.

$i = 4, S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\} : \vec{v_4} \in \text{span}(S)$. We can make $\vec{v_4}$ using $-v_1 + -v_2 + 4\vec{v_3}$.

Therefore, $\boxed{S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}}$. This is a set of 3 vectors. There are other possible answers too, like $S = \{\vec{v_1}, \vec{v_2}, \vec{v_4}\}$; all possible answers have 3 vectors. In fact, since $\text{span}(\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}) = \mathbb{R}^3$, any three vectors in $\mathbb{R}^3$ that don't all lie on the same plane will form a linearly independent set with the same span as $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

b)

$$\vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v_4} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v_5} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v_6} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Hint: Use the 0's in the vectors strategically, plus use the fact that you can't have more than 4 linearly independent vectors in $\mathbb{R}^4$.

Solution:

$$\begin{aligned} \text{Start with } S &= \{\vec{v_1}\} \\ i = 2 &: \vec{v_2} \notin \text{span}(S), \;\; S = \{\vec{v_1}, \vec{v_2}\} \\ i = 3 &: \vec{v_3} \notin \text{span}(S), \;\; S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\} \\ i = 4 &: \vec{v_4} \in \text{span}(S), \;\; -\vec{v_1} + \vec{v_2} = \vec{v_4} \\ i = 5 &: \vec{v_5} \in \text{span}(S), \;\; -\vec{v_1} + \vec{v_3} = \vec{v_5} \\ i = 6 &: \vec{v_6} \in \text{span}(S), \;\; -\vec{v_2} + \vec{v_3} = \vec{v_6} \end{aligned}$$

$$\text{Therefore: } \boxed{S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}}.$$

This is a set of 3 vectors.

Activity 5: Planes and the Cross Product

An important idea from Chapter 4.1 is that two non-parallel vectors in $\mathbb{R}^n$ (where $n \geq 2$) span a plane in n-dimensional space. Here, we'll show you how to find the equation of such a plane, given two vectors in $\mathbb{R}^3$. This is also touched on in Chapter 4.4.

a) Given two vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$, show that the vector $\vec{q}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

$$\vec{q} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

The vector $\vec{q}$ is called the **cross product** of $\vec{a}$ and $\vec{b}$. The cross product is only defined for two vectors in $\mathbb{R}^3$ specifically, and the product is another vector in $\mathbb{R}^3$. (This differentiates it from the dot product, which is defined for two vectors in any $\mathbb{R}^n$, and whose output is a scalar.)

Solution:

$$\begin{aligned}\vec{a}\cdot\vec{q} &= a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) \\ &= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_1a_2b_3 + a_1a_3b_2 - a_2a_3b_1 \\ &= 0\end{aligned}$$

$$\begin{aligned}\vec{b}\cdot\vec{q} &= b_1(a_2b_3 - a_3b_2) + b_2(a_3b_1 - a_1b_3) + b_3(a_1b_2 - a_2b_1) \\ &= a_2b_1b_3 - a_3b_1b_2 + a_3b_1b_2 - a_1b_2b_3 + a_1b_2b_3 - a_2b_1b_3 \\ &= 0\end{aligned}$$

b) Find the cross product of $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

Solution:
Let's call the cross product of $\vec{v}_1$ and $\vec{v}_2$ $\vec{q}$.

$$\begin{aligned}\vec{q} &= \begin{bmatrix} (-1)\cdot(-1) - 3\cdot 2 \\ 3\cdot 1 - 2\cdot(-1) \\ 2\cdot 2 - (-1)\cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-6 \\ 3+2 \\ 4+1 \end{bmatrix} \\ &= \begin{bmatrix} -5 \\ 5 \\ 5 \end{bmatrix}\end{aligned}$$

c) Let $\vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$ be your answer to part **b)**.

Verify that the points $(0,0,0)$, $(2,-1,3)$ and $(1,2,-1)$ satisfy the equation

$$q_1x + q_2y + q_3z = 0$$

(Those points are the endpoints of the vectors $\vec{v}_1$ and $\vec{v}_2$, along with the origin.)

Solution:
The equation of the plane in question is

$$-5x + 5y + 5z = 0$$

For $(0, 0, 0)$,

$$-5(0) + 5(0) + 5(0) = 0$$

For $(2, -1, 3)$,

$$\begin{aligned} & -5(2) + 5(-1) + 5(3) \\ & = -10 - 5 + 15 \\ & = 0 \end{aligned}$$

For $(1, 2, -1)$,

$$\begin{aligned} & -5(1) + 5(2) + 5(-1) \\ & = -5 + 10 - 5 \\ & = 0 \end{aligned}$$

Note that a simpler way of writing the equation of the plane is $x - y - z = 0$, which comes from dividing both sides of the original equation by -5. Another equivalent form is $z = x - y$.

d) Above, we wrote the equation of the plane spanned by $\vec{v}_1$ and $\vec{v}_2$ in the "standard form" for planes in $\mathbb{R}^3$, $ax + by + cz + d = 0$ (where $d = 0$). Now, write the equation of the plane spanned by $\vec{v}_1$ and $\vec{v}_2$ in **parametric** form. The parametric form of a plane is given by

$$P = \vec{p}_0 + s\vec{u} + t\vec{v}, \quad s, t \in \mathbb{R}$$

This won't require much work; it's more that we want you to understand that there are two ways of expressing planes in $\mathbb{R}^3$. In higher dimensions, all planes (also called 2-dimensional subspaces) must be expressed in parametric form. Read Chapter 4.4.

Solution: $P = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad s_1, s_2 \in \mathbb{R}$

The plane spanned by $\vec{v_1}$ and $\vec{v_2}$ is the set of all of their possible linear combinations, shown through the scalars s_1 and s_2. The $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ at the start isn't necessary to write, since if we set $s_1 = s_2 = 0$ we get the point $(0, 0, 0)$ anyways, but it's good to emphasize that the span of $\vec{v_1}$ and $\vec{v_2}$ includes the origin. **The span of any collection of vectors always includes the origin.**

The rest of this worksheet is extra practice. Don't feel pressured to answer all of these problems in lab, but make sure to attempt them at some point.

Activity 6: Projections and Norms

a) Suppose $\vec{u}$ and $\vec{v}$ are two **unit vectors** in $\mathbb{R}^n$ — meaning that $\|\vec{u}\| = 1$ and $\|\vec{v}\| = 1$ — and that the angle between them is α.

Show that the projection of $\vec{u}$ onto $\vec{v}$ is $\vec{p} = (\cos\alpha)\,\vec{v}$, and that the projection of $\vec{v}$ onto $\vec{u}$ is $\vec{q} = (\cos\alpha)\,\vec{u}$.

Solution:
Since $\vec{u}$ and $\vec{v}$ are unit vectors, we know that $\|\vec{u}\| = 1$ and $\|\vec{v}\| = 1$.

$$\begin{aligned}
\vec{p} &= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} \\
&= \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} \\
&= (\vec{u}\cdot\vec{v})\vec{v} && (\|\vec{v}\|^2 = 1) \\
&= (\|\vec{u}\|\|\vec{v}\|\cos\alpha)\vec{v} \\
&= (\cos\alpha)\vec{v}
\end{aligned}$$

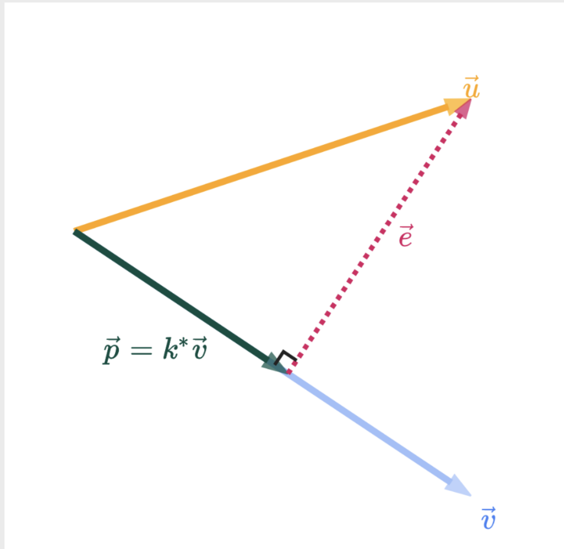
$$\begin{aligned}
\vec{q} &= \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2}\vec{u} \\
&= (\vec{u}\cdot\vec{v})\vec{u} \\
&= (\|\vec{u}\|\|\vec{v}\|\cos\alpha)\cdot\vec{u} \\
&= (\cos\alpha)\vec{u}
\end{aligned}$$

The key ideas here are that if the two vectors we're dealing with are unit vectors, then (1) the dot product of the two vectors is the cosine of the angle between the two vectors, and (2) the projection of one vector onto the other is simply the other vector multiplied by the cosine of the angle between the two vectors.

b) Now, suppose that $\vec{u}, \vec{v} \in \mathbb{R}^n$ are arbitrary vectors, not necessarily unit vectors, and suppose $\vec{p}$ is the projection of $\vec{u}$ onto $\vec{v}$.

- Is it possible for $\vec{p}$ to be longer than $\vec{u}$? If so, give an example. If not, prove why not.
- Is it possible for $\vec{p}$ to be longer than $\vec{v}$? If so, give an example. If not, prove why not.

Solution: **It's not possible for $\vec{p}$ to be longer than $\vec{u}$**, because $\vec{p}$, $\vec{u}$, and the error vector $\vec{u} - \vec{p}$ form a right triangle, where $\vec{u}$ is the hypotenuse. So $\vec{u}$ must be the longest of the three.



It is possible for $\vec{p}$ to be longer than $\vec{v}$. Remember that $\vec{p}$ is just a scalar multiple of $\vec{v}$, where the scalar is

$$\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

There's nothing stopping this scalar from being greater than 1 (or less than -1), which would make $\vec{p}$ longer than $\vec{v}$. As a concrete example, consider $\vec{u} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then,

$$\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} = \frac{3 \cdot 1 + 3 \cdot 0}{1^2 + 0^2} = 3$$

So, $\vec{p} = 3\vec{v}$, which is longer than $\vec{v}$.

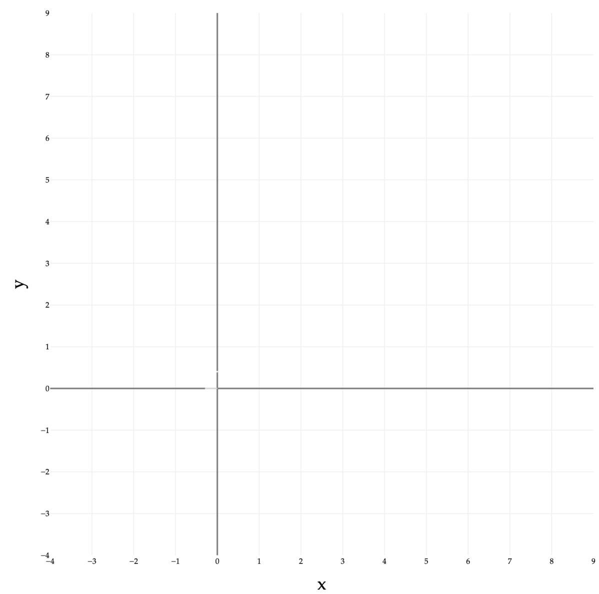
Activity 7: Lines in $\mathbb{R}^n$

A line in $\mathbb{R}^n$ can be expressed in **parametric** form as

$$L = \vec{p}_0 + t\vec{v}, t \in \mathbb{R}$$

where $\vec{p}$ is a point on the line, and $\vec{v}$ is a vector that points in the direction of the line. t is a free variable; different values for t will give us different points on L. The more formal way of stating this definition is $L = \{\vec{p}_0 + t\vec{v} \mid t \in \mathbb{R}\}$.

a) On the grid below, draw the vector $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, the vector $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$, and the line $L = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 6 \\ -3 \end{bmatrix}, t \in \mathbb{R}$.



Solution: We're asked to draw the vectors and the line

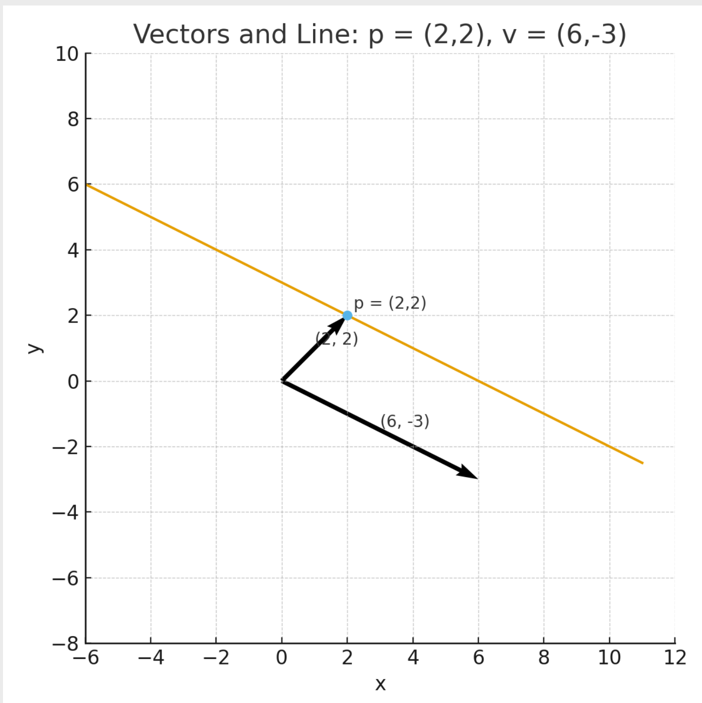
$$L = \vec{p} + t\vec{v}, \quad t \in \mathbb{R}$$

where $\vec{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$.

- The vector $\vec{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is plotted from the origin. This "fixes" the point $(2, 2)$ on the line.
- The direction vector $\vec{v} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$ is drawn from the origin. This vector indicates the slope and direction of the line.
- To sketch the line L, we start at the point $\vec{p}$ and extend in the direction of $\vec{v}$. Let's plug in some values of t to get some points on the line:

  $$t = 0 \;\Rightarrow\; (2, 2) \qquad t = 1 \;\Rightarrow\; (2, 2) + (6, -3) = (8, -1)$$

  Plotting these points and connecting them with a straight line shows L. Extending the line in both directions corresponds to varying t over all real numbers.



b) Express the line $L = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 6 \\ -3 \end{bmatrix}, t \in \mathbb{R}$ in the "standard" form for lines in $\mathbb{R}^2$, $y = mx + b$. (Remember that only lines in $\mathbb{R}^2$ can be expressed in this form; in higher dimensions, we need to use the parametric form. Think about why this is the case, and consult **Chapter 4.4**.)

Solution:
We want to convert the parametric line

$$L = \vec{p} + t\vec{v}, \quad t \in \mathbb{R} \quad \text{where} \quad \vec{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

into the slope–intercept form $y = mx + b$. There are a few possible solutions.

Solution 1: Solve for t in terms of x
One solution is to write out the two components of the line as two separate equations:

$$x = 2 + 6t$$
$$y = 2 - 3t$$

These give the x- and y-coordinates of a point on L as t varies.

Let's solve the first equation for t, and then substitute that into the second equation to get y as a function of x.

$$t = \frac{x-2}{6}$$
$$\implies y = 2 - 3\left(\frac{x-2}{6}\right) = 2 - \frac{1}{2}(x-2) = 2 - \frac{1}{2}x + 1 = -\frac{1}{2}x + 3$$

Thus,

$$\boxed{y = -\tfrac{1}{2}x + 3}$$

Solution 2: Find two points on the line
Another solution is to pick two points off the line and use them to find the slope and intercept. As we saw in part **a)**, the points $(2, 2)$ and $(8, -1)$ are both on the line, so its slope is

$$m = \frac{-1-2}{8-2} = -\frac{1}{2}$$

and its intercept is

$$b = 2 - m \cdot 2 = 2 - (-\frac{1}{2}) \cdot 2 = 3$$

So, the line is

$$y = -\frac{1}{2}x + 3$$

You can also more easily find the slope by looking at the direction vector $\vec{v} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$, which says that for every 6 units we move to the right, we move 3 units down, which implies a slope of $\frac{-3}{6} = -\frac{1}{2}$.

c) Why is the line $L = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 6 \\ -3 \end{bmatrix}, t \in \mathbb{R}$ **not** equal to the span of any one vector in $\mathbb{R}^2$?

Solution:
As we saw in **Chapter 4.4**, a key fact about the span of a single vector is that it is always a line through the origin. That is, for any $\vec{v} \in \mathbb{R}^n$,

$$\text{span}(\{\vec{v}\}) = \{s\vec{v} \mid s \in \mathbb{R}\}$$

If $s = 0$ above, we get the point $(0, 0, ..., 0)$, which must be in the span of any individual vector.

So, the short answer is that L is not the span of any single vector because it doesn't contain the origin. It doesn't have the origin as a fixed point — the fixed point it's defined in terms of is $(2, 2)$ — but we can also show that there is no t such that $\begin{bmatrix} 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
If $(0, 0) \in L$, there must be $t \in \mathbb{R}$ with

$$0 = 2 + 6t$$

$$0 = 2 - 3t$$

The first equation implies that $t = -\frac{1}{3}$, and the second equation implies that $t = \frac{2}{3}$. These values are inconsistent, so $(0, 0) \notin L$, and L **cannot be the span of a single vector**.

d) Find a line in $\mathbb{R}^4$ that passes through $(0, 1, 2, 3)$ and is **orthogonal** to $\begin{bmatrix} 9 \\ 3 \\ 1 \\ -5 \end{bmatrix}$.

Solution:
Our line will be of the form

$$L = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}}_{\text{fixed point}} + t\vec{v}, \quad t \in \mathbb{R}$$

where $\vec{v}$ is the vector that describes the direction of the line. We want our line to be orthogonal to $\begin{bmatrix} 9 \\ 3 \\ 1 \\ -5 \end{bmatrix}$, so we need to find a vector $\vec{v}$ that is orthogonal to $\begin{bmatrix} 9 \\ 3 \\ 1 \\ -5 \end{bmatrix}$.

$$\begin{bmatrix} 9 \\ 3 \\ 1 \\ -5 \end{bmatrix} \cdot \vec{v} = 0 \implies 9v_1 + 3v_2 + v_3 - 5v_4 = 0$$

There are infinitely many solutions for v_1, v_2, v_3, v_4 that satisfy the equation, meaning there are infinitely many possible lines that satisfy the condition in the question. Let's pick one direction vector: $v_1 = 1$, $v_2 = -4$, $v_3 = 3$, $v_4 = 0$. This gives

$$9(1) + 3(-4) + 1(3) + (-5)(0) = 9 - 12 + 3 + 0 = 0$$

So, $\vec{v} = \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}$ is a valid direction vector for our line, and one possible line is

$$L = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

Activity 8: A Plane that Doesn't Pass Through the Origin

In $\mathbb{R}^2$, any two points uniquely determine a line. In $\mathbb{R}^3$, any three points uniquely determine a plane.

Consider the points $(3, 4, 5)$, $(1, 9, -2)$, and $(2, 2, 0)$. Find the equation of the plane that passes through all three points, and express that plane in (1) parametric form and (2) standard form,

$$ax + by + cz + d = 0$$

Hint: The plane does not necessarily pass through the origin, unlike the plane we found in part ***c)****, which had to pass through the origin by virtue of being the span of a set of vectors.*

The parametric form is much easier to find — start with it, and use your answer to find the equation in standard form. Think about how the cross product might apply. Then, think about how to find the offset, d.

Solution: Start by picking one of the three points; we'll arbitrarilychoose $(3, 4, 5)$ for this solution. We now need to find two vectors on the plane, which we can do by subtracting our chosen point from the other two.

$$\vec{u} = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} -2 \\ 5 \\ -7 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ -2 \\ -5 \end{bmatrix}$$

Now that we have our vectors, we can write the plane in parametric form:

$$P = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + s_1 \begin{bmatrix} -2 \\ 5 \\ -7 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ -2 \\ -5 \end{bmatrix}, \quad s_1, s_2 \in \mathbb{R}$$

In part **c)**, we used the components of the cross product to define the equation of the plane spanned by two vectors, so let's calculate that next:

$$c = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$
$$= \begin{bmatrix} 5 \cdot (-5) - (-7) \cdot (-2) \\ (-7) \cdot (-1) - (-2) \cdot (-5) \\ (-2) \cdot (-2) - 5 \cdot (-1) \end{bmatrix}$$
$$= \begin{bmatrix} -25 - 14 \\ 7 - 10 \\ 4 + 5 \end{bmatrix}$$
$$= \begin{bmatrix} -39 \\ -3 \\ 9 \end{bmatrix}$$

We've got one last step, and that's to solve for the offset d in $ax + by + cz + d = 0$. We can do this by plugging any of our points into the equation with our vector components as coefficients.

$$\begin{aligned}
-39x - 3y + 9z + d &= 0 \\
-39(3) - 3(4) + 9(5) + d &= 0 \\
-117 - 12 + 45 + d &= 0 \\
-84 + d &= 0 \\
d &= 84
\end{aligned}$$

So, our final equation is

$$-39x - 3y + 9z + 84 = 0$$

or equivalently,

$$13x + y - 3z - 28 = 0$$

You can verify that the points $(3, 4, 5)$, $(1, 9, -2)$, and $(2, 2, 0)$ all satisfy the equation above.