Lab 5: Projections and Spans

EECS 245, Winter 2026 at the University of Michigan

due by the end of your lab section

Name: ____________________

uniqname: ____________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

Recap: Projections and Spans

- (Chapter 3.4) The **orthogonal projection** of the vector $\vec{u}$ onto the vector $\vec{v}$ is given by

$$\vec{p} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$

  Above, the scalar $k^* = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$ was chosen to minimize $\|\vec{u} - k\vec{v}\|^2$.

- The vector $\vec{p}$ is called the orthogonal projection because the resulting error vector,

$$\vec{e} = \vec{u} - \vec{p} = \vec{u} - k^* \vec{v}$$

  is orthogonal to $\vec{v}$.

$$\vec{e} \cdot \vec{v} = 0$$

- (4.1) The **span** of a set of vectors is the set of all possible linear combinations of the vectors in the set.

$$\text{span}(\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d\}) = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_d\vec{v}_d \mid a_1, a_2, \dots, a_d \in \mathbb{R}\}$$

  - The span of one vector in $\mathbb{R}^n$ is a line through the origin.
  - The span of two **non-parallel** vectors in $\mathbb{R}^n$ is a plane through the origin; this plane is called a **2-dimensional subspace** of $\mathbb{R}^n$.
  - In general, the span of d vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ of dimension 0 to d, depending on the vectors and their relationships.
  - Think of a d-dimensional subspace of $\mathbb{R}^n$ as a "slice" of $\mathbb{R}^n$ that goes through the origin, in which you can move in d directions.

Activity 1: Presidential Speeches and Cosine Similarity

Complete the tasks in the `lab05.ipynb` notebook, which you can either access through the DataHub link on the course homepage or by pulling our GitHub repository. To receive credit for Activity 1, you'll need to show your lab TA that all test cases have passed. Instructions on how to do this are in the lab notebook.

Activity 2: Orthogonal Projections

Let $\vec{c} = \begin{bmatrix} 1 \\ 2 \\ -4 \\ 0 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix}$. Note that $\|\vec{c}\|^2 = 21$, $\|\vec{d}\|^2 = 14$, and $\vec{c} \cdot \vec{d} = 7$.

a) Find the orthogonal projection of $\vec{c}$ onto $\vec{d}$. Call this vector $\vec{q}$.

b) Find the error vector, $\vec{r} = \vec{c} - \vec{q}$. Which vector is $\vec{r}$ orthogonal to, $\vec{c}$ or $\vec{d}$? Draw a rough picture of the relationship between $\vec{c}$, $\vec{d}$, $\vec{q}$, and $\vec{r}$.

Activity 3: Orthogonal Decomposition

a) Let $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Write $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, and verify that your answer is correct. Note that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are pairwise orthogonal.

b) In general, suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are **orthogonal** vectors in $\mathbb{R}^n$, meaning that $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$. **Given that** it is possible to write $\vec{u}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$, show that the coefficients of the linear combination

$$\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_d\vec{v}_d$$

are given by

$$a_i = \frac{\vec{u} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

Hint: Start by taking the dot product of both sides of the linear combination equation with $\vec{v}_i$. What do you notice?

Activity 4: Finding a Linearly Independent Subset

(This problem also appears as the final, optional problem in Homework 4. It is meant to prepare you for Problem 4 on Homework 4, which asks a very similar question.)

Recall from Chapter 4.2 that a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ is **linearly independent** if either of the following equivalent conditions hold:

- None of the vectors can be written as a linear combination of the others.
- The only way to create the zero vector as a linear combination of the vectors is if all the coefficients are zero. In other words, the only solution to

  $$a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d = \vec{0}$$

  is $a_1 = a_2 = \ldots = a_d = 0$.

Chapter 4.2 introduces an algorithm for finding a linearly independent subset of a given set of vectors with the same span as the original set:

```
given v_1, v_2, ..., v_d
initialize linearly independent set S = {v_1}
for i = 2 to d:
    if v_i is not a linear combination of S:
        add v_i to S
```

In each of the parts below, find **a linearly independent** set of vectors that spans the same span as the given set of vectors. There are multiple possible answers for each part, but all of them have the same number of vectors.

a)

$$\vec{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix}2\\3\\4\end{bmatrix}$$

b)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Hint: Use the 0's in the vectors strategically, plus use the fact that you can't have more than 4 linearly independent vectors in $\mathbb{R}^4$.

Activity 5: Planes and the Cross Product

An important idea from **Chapter 4.1** is that two non-parallel vectors in $\mathbb{R}^n$ (where $n \geq 2$) span a plane in n-dimensional space. Here, we'll show you how to find the equation of such a plane, given two vectors in $\mathbb{R}^3$. This is also touched on in **Chapter 4.4**.

a) Given two vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$, show that the vector $\vec{q}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

$$\vec{q} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

The vector $\vec{q}$ is called the **cross product** of $\vec{a}$ and $\vec{b}$. The cross product is only defined for two vectors in $\mathbb{R}^3$ specifically, and the product is another vector in $\mathbb{R}^3$. (This differentiates it from the dot product, which is defined for two vectors in any $\mathbb{R}^n$, and whose output is a scalar.)

b) Find the cross product of $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

c) Let $\vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$ be your answer to part **b)**.

Verify that the points $(0,0,0)$, $(2,-1,3)$ and $(1,2,-1)$ satisfy the equation

$$q_1 x + q_2 y + q_3 z = 0$$

(Those points are the endpoints of the vectors $\vec{v}_1$ and $\vec{v}_2$, along with the origin.)

d) Above, we wrote the equation of the plane spanned by $\vec{v}_1$ and $\vec{v}_2$ in the "standard form" for planes in $\mathbb{R}^3$, $ax + by + cz + d = 0$ (where $d = 0$). Now, write the equation of the plane spanned by $\vec{v}_1$ and $\vec{v}_2$ in **parametric** form. The parametric form of a plane is given by

$$P = \vec{p}_0 + s\vec{u} + t\vec{v}, \quad s, t \in \mathbb{R}$$

This won't require much work; it's more that we want you to understand that there are two ways of expressing planes in $\mathbb{R}^3$. In higher dimensions, all planes (also called 2-dimensional subspaces) must be expressed in parametric form. Read **Chapter 4.4**.

The rest of this worksheet is extra practice. Don't feel pressured to answer all of these problems in lab, but make sure to attempt them at some point.

Activity 6: Projections and Norms

a) Suppose $\vec{u}$ and $\vec{v}$ are two **unit vectors** in $\mathbb{R}^n$ — meaning that $\|\vec{u}\| = 1$ and $\|\vec{v}\| = 1$ — and that the angle between them is α.

Show that the projection of $\vec{u}$ onto $\vec{v}$ is $\vec{p} = (\cos\alpha)\,\vec{v}$, and that the projection of $\vec{v}$ onto $\vec{u}$ is $\vec{q} = (\cos\alpha)\,\vec{u}$.

b) Now, suppose that $\vec{u}, \vec{v} \in \mathbb{R}^n$ are arbitrary vectors, not necessarily unit vectors, and suppose $\vec{p}$ is the projection of $\vec{u}$ onto $\vec{v}$.

- Is it possible for $\vec{p}$ to be longer than $\vec{u}$? If so, give an example. If not, prove why not.
- Is it possible for $\vec{p}$ to be longer than $\vec{v}$? If so, give an example. If not, prove why not.

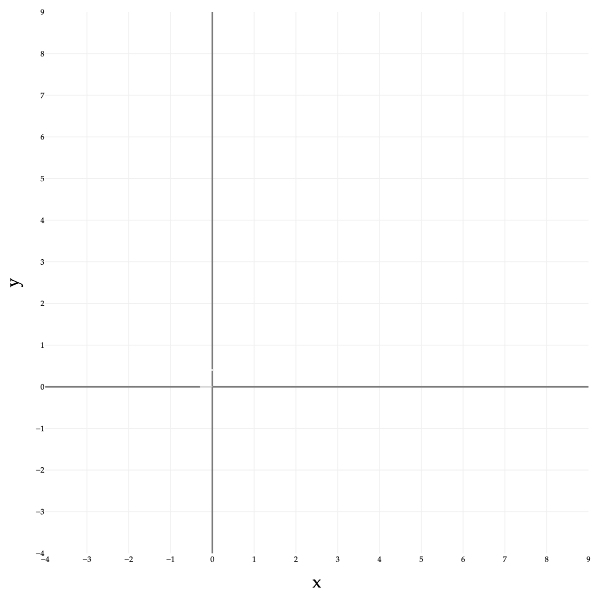
Activity 7: Lines in $\mathbb{R}^n$

A line in $\mathbb{R}^n$ can be expressed in **parametric** form as

$$L = \vec{p}_0 + t\vec{v}, t \in \mathbb{R}$$

where $\vec{p}$ is a point on the line, and $\vec{v}$ is a vector that points in the direction of the line. t is a free variable; different values for t will give us different points on L. The more formal way of stating this definition is $L = \{\vec{p}_0 + t\vec{v} \mid t \in \mathbb{R}\}$.

a) On the grid below, draw the vector $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, the vector $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$, and the line $L = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + t\begin{bmatrix} 6 \\ -3 \end{bmatrix}, t \in \mathbb{R}$.



b) Express the line $L = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + t\begin{bmatrix} 6 \\ -3 \end{bmatrix}, t \in \mathbb{R}$ in the "standard" form for lines in $\mathbb{R}^2$, $y = mx + b$. (Remember that only lines in $\mathbb{R}^2$ can be expressed in this form; in higher dimensions, we need to use the parametric form. Think about why this is the case, and consult Chapter 4.4.)

c) Why is the line $L = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + t\begin{bmatrix} 6 \\ -3 \end{bmatrix}, t \in \mathbb{R}$ **not** equal to the span of any one vector in $\mathbb{R}^2$?

d) Find a line in $\mathbb{R}^4$ that passes through $(0, 1, 2, 3)$ and is **orthogonal** to $\begin{bmatrix} 9 \\ 3 \\ 1 \\ -5 \end{bmatrix}$.

Activity 8: A Plane that Doesn't Pass Through the Origin

In $\mathbb{R}^2$, any two points uniquely determine a line. In $\mathbb{R}^3$, any three points uniquely determine a plane.

Consider the points $(3, 4, 5)$, $(1, 9, -2)$, and $(2, 2, 0)$. Find the equation of the plane that passes through all three points, and express that plane in (1) parametric form and (2) standard form,

$$ax + by + cz + d = 0$$

Hint: The plane does not necessarily pass through the origin, unlike the plane we found in part ***c)****, which had to pass through the origin by virtue of being the span of a set of vectors.*

The parametric form is much easier to find — start with it, and use your answer to find the equation in standard form. Think about how the cross product might apply. Then, think about how to find the offset, d.