Lab 5: Vector Spaces, Subspaces, and Bases

EECS 245, Fall 2025 at the University of Michigan

due by the end of your lab section on Wednesday, September 24th, 2025

Name: ______________________________

uniqname: ______________________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

Acknowledgements: Activities 1, 3, and 6 are taken from **here**, and Activity 4 is taken from *Linear Algebra* by Gilbert Strang. Consider looking at these sources for more practice problems.

Activity 1: Linear Independence

Let $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

a) Find scalars a, b, c, and d such that $a\vec{w} + b\vec{x} + c\vec{y} + d\vec{z} = \vec{0}$, and at least one of the scalars is non-zero. By doing so, you're showing that $\vec{w}, \vec{x}, \vec{y}, \vec{z}$ are linearly dependent.

b) Find scalars A, B, and C such that $\vec{z} = A\vec{w} + B\vec{x} + C\vec{y}$. This is another way of showing that $\vec{w}, \vec{x}, \vec{y}, \vec{z}$ are linearly dependent.

c) Show that $\text{span}(\{\vec{w}, \vec{x}, \vec{y}, \vec{z}\}) \neq \mathbb{R}^4$ by finding a vector $\vec{v} \in \mathbb{R}^4$ such that $\vec{v} \notin \text{span}\{\vec{w}, \vec{x}, \vec{y}, \vec{z}\}$.

d) Why is the fact that $\text{span}(\{\vec{w}, \vec{x}, \vec{y}, \vec{z}\}) \neq \mathbb{R}^4$ enough to conclude that $\vec{w}, \vec{x}, \vec{y}, \vec{z}$ are linearly **dependent**?

Activity 2: Formal Definition of Linear Independence

Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d \in \mathbb{R}^n$, and that $\vec{b} \in \text{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\})$.

a) Give a one sentence English explanation of what it means for $\vec{b} \in \text{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\})$.

b) Suppose that $a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d = \vec{b}$ **and** $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_d\vec{v}_d = \vec{b}$, where at least one of the a_i's is different from its corresponding c_i.
Using the formal definition of linear independence from Chapter 2.4, determine whether or not $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are linearly independent, and prove your answer.

c) Find another set of coefficients $k_1, k_2, \ldots, k_d$ such that

$$k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots + k_d\vec{v}_d = \vec{b}$$

and at least one of the k_i's is different from its corresponding a_i or c_i.

By doing this, you're showing that if there is at least one way to write $\vec{b}$ as a linear combination of a set of vectors, then there are infinitely many ways to write $\vec{b}$ as a linear combination of those vectors; there can't just be two or three ways to do it.

Activity 3: Introduction to Subspaces

As discussed in Chapter 2.6, a **subspace** S of a vector space V is a subset of V that itself is a vector space, contains the zero vector, and is **closed** under addition and scalar multiplication. That is, if you take any two vectors in in S, any of their linear combinations must also be in S.

Only one of the following is a subspace of $\mathbb{R}^3$. Which one? Explain why the others are not subspaces.

The set of vectors $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ such that

(i) $x + 2y - 3z = 4$

(ii) $\vec{v}$ is on the line $L = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, t \in \mathbb{R}$

(iii) $x + y + z = 0$ and $x - y + z = 1$

(iv) $x = -z$ and $x = z$

(v) $x^2 + y^2 = z$

Activity 4: Finding Non-Examples of Subspaces

In this activity, you'll find sets of vectors in $\mathbb{R}^2$ that satisfy some, but not all, of the requirements for a subspace. Think creatively, and since we're working in $\mathbb{R}^2$, visualize the vectors!

a) Find a set of vectors in $\mathbb{R}^2$ such that the sum of any two vectors $\vec{u}$ and $\vec{v}$ in the set is also in the set, but $\frac{1}{2}\vec{v}$ is possibly not in the set.

b) Find a set of vectors in $\mathbb{R}^2$ such that $c\vec{v}$ is in the set for any vector $\vec{v}$ in the set and any scalar c, but the sum of any two vectors $\vec{u}$ and $\vec{v}$ in the set is possibly not in the set.

Activity 5: Bases

Recall from Chapter 2.6 that a **basis** for a subspace S is a set of vectors that

1. span all of S, **and**
2. are linearly independent

In each part below, find **two different possible bases** for the given vector space, and state the **dimension** of the vector space. (Note that this is effectively what you're doing in **Problems 4 and 5 of Homework 4**, we just hadn't introduced the term "basis" at that point.)

a) $S = \operatorname{span}\left(\left\{\begin{bmatrix}1\\3\\3\end{bmatrix}, \begin{bmatrix}-3\\-9\\-9\end{bmatrix}, \begin{bmatrix}1\\5\\-1\end{bmatrix}, \begin{bmatrix}2\\7\\4\end{bmatrix}, \begin{bmatrix}1\\4\\1\end{bmatrix}\right\}\right)$

b) $S = \left\{\begin{bmatrix}v_1\\v_2\end{bmatrix} \mid v_1 = -v_2; v_1, v_2 \in \mathbb{R}\right\}$

c) $S = \left\{\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} \mid v_4 = 0; v_1, v_2, v_3 \in \mathbb{R}\right\}$

Activity 6: Intersections of Subspaces

Let:

- M be the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix}1\\1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\-4\\1\\5\end{bmatrix}$
- N be the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix}0\\-2\\1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\-1\\1\\3\end{bmatrix}$

a) Find a vector that belongs to both M and N. (In other words, find a vector $\vec{v}$ such that $\vec{v} \in M$ and $\vec{v} \in N$.)

There are infinitely many answers; pick the answer with a first component of 1.

b) Fill in the blanks: the set of all vectors that belong to both M and N is a subspace of $\mathbb{R}^4$ with dimension ___________.

Use the space below for scratch work.