Lab 6: Rank, Column Space, and Null Space Solutions

EECS 245, Fall 2025 at the University of Michigan

due by the end of your lab section on Wednesday, October 8th, 2025

Name: ______________________________

uniqname: ______________________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

This worksheet involves reinforcing ideas from **Chapter 2.8**. To briefly review, suppose A is an $n \times d$ matrix. Recall that $\text{rank}(A)$ is the number of linearly independent columns of A.

	Notation	Description	Dimension	Subspace of
Column space	$\text{colsp}(A)$	Span of the columns of A	$\text{rank}(A)$	$\mathbb{R}^n$
Row space	$\text{colsp}(A^T)$	Span of the rows of A	$\text{rank}(A)$	$\mathbb{R}^d$
Nullspace	$\text{nullsp}(A)$	Set of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$	$d - \text{rank}(A)$	$\mathbb{R}^d$

Activity 1: Programming (Do this at the end of the lab, not the beginning!)

Complete the tasks in the `lab06.ipynb` notebook, which you can either access through the DataHub link on the course homepage or by pulling our GitHub repository. To receive credit for Activity 1, show your lab TA that you've passed all test cases in your notebook; **there's no need to submit your notebook to Gradescope**.

Activity 2: Null Space of a Matrix with Linearly Independent Columns

Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \\ 1 & 0 \end{bmatrix}$. What is $\text{nullsp}(A)$?

Solution:
We are asked to find the null space of

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \\ 1 & 0 \end{bmatrix}$$

The null space of A, denoted nullsp(A), is the set of all vectors $\vec{x}$ such that

$$A\vec{x} = \vec{0}$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ Then:

$$A\vec{x} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 4x_2 \\ x_1 \end{bmatrix}$$

$$\begin{bmatrix} 3x_1 \\ 4x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives the system of equations:

$$\begin{aligned} 3x_1 &= 0 \\ 4x_2 &= 0 \\ x_1 &= 0 \end{aligned}$$

$$x_1 = 0 \quad x_2 = 0$$

The only vector that satisfies this system is

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{nullsp}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \{\vec{0}\}$$

Since A has two linearly independent columns, its null space contains only the zero vector. This means A has full column rank and no nontrivial solutions to $A\vec{x} = \vec{0}$.

Activity 3: Fundamentals

Let $X = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 & 4 \\ 2 & 5 & -2 & 7 & 11 & 10 \\ 4 & 8 & -4 & 12 & 16 & 16 \end{bmatrix}$.

a) Find a basis for colsp(X). What is rank(X)? Why?

Solution:
We are given

$$X = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 & 4 \\ 2 & 5 & -2 & 7 & 11 & 10 \\ 4 & 8 & -4 & 12 & 16 & 16 \end{bmatrix}$$

Let the columns of X be

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -1 \\ -2 \\ -4 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 3 \\ 7 \\ 12 \end{bmatrix} \quad \vec{v}_5 = \begin{bmatrix} 4 \\ 11 \\ 16 \end{bmatrix} \quad \vec{v}_6 = \begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix}$$

We will use the algorithm

> Given $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$
> Initialize $S = \{\vec{v}_1\}$
> For $i = 2, \ldots, d$
> If $\vec{v}_i$ is not a linear combination of S, add $\vec{v}_i$ to S

Start with $S = \{\vec{v}_1\}$

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\}$$

Check if $\vec{v}_2$ is a linear combination of S
If $\vec{v}_2 = c\vec{v}_1$, then comparing entries gives

$$\frac{2}{1} = 2 \quad \frac{5}{2} = 2.5 \quad \frac{8}{4} = 2$$

The ratios are not equal, so $\vec{v}_2$ is not a multiple of $\vec{v}_1$
Add $\vec{v}_2$ to S

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\}$$

Check if $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$
We seek a, b such that

$$a\vec{v}_1 + b\vec{v}_2 = \vec{v}_3$$

$$a \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + b \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -4 \end{bmatrix}$$

This gives the system

$$\begin{cases} a + 2b = -1 \\ 2a + 5b = -2 \\ 4a + 8b = -4 \end{cases}$$

From the first two equations, subtract $2\times$(first) from the second:

$$(2a + 5b) - (2a + 4b) = -2 - (-2) = 0 \Rightarrow b = 0$$

Then $a = -1$ This satisfies all three equations, so $\vec{v}_3 = -1\vec{v}_1$ Hence $\vec{v}_3$ is a linear combination of S and is not added.

Check $\vec{v}_4$

$$a\vec{v}_1 + b\vec{v}_2 = \vec{v}_4 \Rightarrow a\begin{bmatrix}1\\2\\4\end{bmatrix} + b\begin{bmatrix}2\\5\\8\end{bmatrix} = \begin{bmatrix}3\\7\\12\end{bmatrix}$$

This gives

$$\begin{cases} a + 2b = 3 \\ 2a + 5b = 7 \end{cases}$$

Subtract $2\times$(first) from the second:

$$(2a + 5b) - (2a + 4b) = 7 - 6 \Rightarrow b = 1$$

Then $a = 1$ So $\vec{v}_4 = \vec{v}_1 + \vec{v}_2$ It is a linear combination of S and not added.

Check $\vec{v}_5$

$$a\vec{v}_1 + b\vec{v}_2 = \vec{v}_5 \Rightarrow a\begin{bmatrix}1\\2\\4\end{bmatrix} + b\begin{bmatrix}2\\5\\8\end{bmatrix} = \begin{bmatrix}4\\11\\16\end{bmatrix}$$

$$\begin{cases} a + 2b = 4 \\ 2a + 5b = 11 \end{cases}$$

Subtract $2\times$(first) from the second:

$$(2a + 5b) - (2a + 4b) = 11 - 8 \Rightarrow b = 3$$

Then $a = -2$ Hence $\vec{v}_5 = -2\vec{v}_1 + 3\vec{v}_2$ It is a linear combination of S and not added.

Check $\vec{v}_6$

$$a\vec{v}_1 + b\vec{v}_2 = \vec{v}_6 \Rightarrow a\begin{bmatrix}1\\2\\4\end{bmatrix} + b\begin{bmatrix}2\\5\\8\end{bmatrix} = \begin{bmatrix}4\\10\\16\end{bmatrix}$$

$$\begin{cases} a + 2b = 4 \\ 2a + 5b = 10 \end{cases}$$

Subtract $2\times$(first) from the second:

$$(2a + 5b) - (2a + 4b) = 10 - 8 \Rightarrow b = 2$$

Then $a = 0$ So $\vec{v}_6 = 2\vec{v}_2$ It is a linear combination of S and not added.

At the end of the algorithm, the set S is

$$S = \left\{ \begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix} \right\}$$

$$\boxed{\text{Basis for colsp}(X) = S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\} \qquad \text{rank}(X) = 2}$$

The rank equals the number of linearly independent columns identified by the algorithm.

b) Fill in the blanks: colsp(X^T) is a ____-dimensional subspace of ____.

Solution:
We know that colsp(X^T) is the row space of X and from the previous part rank$(X) = 2$.

The dimension of the row space is equal to the rank of the matrix:

$$\dim(\text{colsp}(X^T)) = \text{rank}(X) = 2$$

The rows of X each lie in $\mathbb{R}^6$ so the row space is a subspace of $\mathbb{R}^6$

$$\boxed{\text{colsp}(X^T) \text{ is a 2-dimensional subspace of } \mathbb{R}^6}$$

c) Fill in the blanks: nullsp(X) is a ____-dimensional subspace of ____.

Solution:
From Chapter 2.8, the **Rank–Nullity Theorem** states that for any $n \times d$ matrix X,

$$\text{rank}(X) + \dim(\text{nullsp}(X)) = d$$

The rank of X was found to be 2, and X has 6 columns, so

$$2 + \dim(\text{nullsp}(X)) = 6$$

$$\dim(\text{nullsp}(X)) = 6 - 2 = 4$$

By definition (see "Null Space" in Section 2.8), nullsp(X) is the set of all vectors $\vec{x} \in \mathbb{R}^6$ such that $X\vec{x} = \vec{0}$. Therefore, it is a subspace of $\mathbb{R}^6$.

$$\boxed{\text{nullsp}(X) \text{ is a 4-dimensional subspace of } \mathbb{R}^6}$$

d) Find a basis for nullsp(X^T).

Solution:
We want to find a basis for nullsp(X^T). Begin by determining the dimension of the null space from the rank of X^T.

$$\text{nullsp}(X^T) = \{\vec{y} \in \mathbb{R}^3 : X^T\vec{y} = \vec{0}\}$$

We have

$$X^T = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \\ -1 & -2 & -4 \\ 3 & 7 & 12 \\ 4 & 11 & 16 \\ 4 & 10 & 16 \end{bmatrix}$$

Each column of X^T corresponds to a row of X. To find nullsp(X^T), we look for all vectors $\vec{y} = [y_1,\, y_2,\, y_3]^T$ such that

$$X^T\vec{y} = \vec{0}$$

First we'll determine rank(X^T), examining the columns of X^T:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 4 \\ 4 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ -2 \\ 7 \\ 11 \\ 10 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 8 \\ -4 \\ 12 \\ 16 \\ 16 \end{bmatrix}$$

We see that $\vec{v}_3$ is a linear combination of the first two columns:

$$\vec{v}_3 = 0 \cdot \vec{v}_1 + 2 \cdot \vec{v}_2 - \vec{v}_1$$

so only $\vec{v}_1$ and $\vec{v}_2$ are linearly independent. Thus,

$$\text{rank}(X^T) = 2$$

Now apply the Rank–Nullity Theorem to find the dimension of the null space.

Since X^T has 3 columns,

$$\text{rank}(X^T) + \dim(\text{nullsp}(X^T)) = 3$$

Substituting rank$(X^T) = 2$ gives

$$\dim(\text{nullsp}(X^T)) = 3 - 2 = 1$$

So the null space is a **1-dimensional line** through the origin in $\mathbb{R}^3$.
Then, using the first two independent rows of X (equivalently, the first two independent equations of X^T),

$$\begin{cases} y_1 + 2y_2 + 4y_3 = 0 \\ 2y_1 + 5y_2 + 8y_3 = 0 \end{cases}$$

Subtracting 2 times the first equation from the second eliminates y_1:

$$(2y_1 - 2y_1) + (5y_2 - 4y_2) + (8y_3 - 8y_3) = 0 \quad \Rightarrow \quad y_2 = 0$$

Substitute $y_2 = 0$ into the first equation:

$$y_1 + 4y_3 = 0 \quad \Rightarrow \quad y_1 = -4y_3$$

Letting $y_3 = 1$ gives one particular solution

$$\vec{y} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

Finally, all solutions are scalar multiples of this vector:

$$\text{nullsp}(X^T) = \text{span}\left(\left\{\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$$

a 1-dimensional subspace of $\mathbb{R}^3$.

$$\boxed{\text{Basis for nullsp}(X^T) = \left\{\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}\right\} \qquad \dim(\text{nullsp}(X^T)) = 1}$$

Suppose A is an $n \times d$ matrix with rank r. A CR decomposition of A is a product of two matrices C and R, where $A = CR$ and:

- C is an $n \times r$ matrix and R is a $r \times d$ matrix
- C contains the linearly independent columns of A, selected left-to-right
- R tells how to "mix" the columns of C (which are linearly independent) to reconstruct the columns of A

Let's keep working with $X = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 & 4 \\ 2 & 5 & -2 & 7 & 11 & 10 \\ 4 & 8 & -4 & 12 & 16 & 16 \end{bmatrix}$.

e) Find a CR decomposition of X. This shouldn't take very much work; review your work from part **a)** in finding a basis for $\text{colsp}(X)$.

Solution:
We are asked to find a CR decomposition of

$$X = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 & 4 \\ 2 & 5 & -2 & 7 & 11 & 10 \\ 4 & 8 & -4 & 12 & 16 & 16 \end{bmatrix}$$

From part **(a)**, we determined that

$$\text{rank}(X) = 2 \quad \text{and} \quad \text{basis for colsp}(X) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\}$$

These two columns of X are linearly independent, and all other columns of X can be written as linear combinations of them.

By definition of the CR decomposition, the matrix C contains the linearly independent columns of X (selected left-to-right). Thus,

$$C = \begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 4 & 8 \end{bmatrix}$$

C has shape 3×2, and its two columns form a basis for colsp(X).

The matrix R describes how to "mix" the columns of C to obtain all six columns of X. That is, each column of R contains the coefficients expressing one column of X as a linear combination of the two columns of C.

We already know the first two columns of X correspond directly to the columns of C:

$$\text{col}_1(X) = 1 \cdot \text{col}_1(C) + 0 \cdot \text{col}_2(C) \quad \text{col}_2(X) = 0 \cdot \text{col}_1(C) + 1 \cdot \text{col}_2(C)$$

So, the first two columns of R are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, we express the remaining columns of X in terms of $\text{col}_1(C)$ and $\text{col}_2(C)$.

For column 3 of X, $\begin{bmatrix} -1 \\ -2 \\ -4 \end{bmatrix}$, we observe it equals $-1 \cdot \text{col}_1(C) + 0 \cdot \text{col}_2(C)$.

For column 4, $\begin{bmatrix} 3 \\ 7 \\ 12 \end{bmatrix}$, we can solve for coefficients a, b in

$$a \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + b \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 12 \end{bmatrix}$$

From the first two entries, $a + 2b = 3$ and $2a + 5b = 7$.

Subtracting twice the first equation from the second gives $b = 1$ and $a = 1$.

So column 4 = $1 \cdot \text{col}_1(C) + 1 \cdot \text{col}_2(C)$.

For column 5, $\begin{bmatrix} 4 \\ 11 \\ 16 \end{bmatrix}$, we solve $a + 2b = 4$ and $2a + 5b = 11$. Subtracting twice the first from the second gives $b = 3$ and $a = -2$. So column 5 = $-2 \cdot \text{col}_1(C) + 3 \cdot \text{col}_2(C)$.

For column 6, $\begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix}$, we solve $a + 2b = 4$ and $2a + 5b = 10$. Subtracting twice the first from the second gives $b = 2$ and $a = 0$. So column 6 = $0 \cdot \text{col}_1(C) + 2 \cdot \text{col}_2(C)$.

$$R = \begin{bmatrix} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 3 & 2 \end{bmatrix}$$

Verifying,

$$CR = \begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 & 4 \\ 2 & 5 & -2 & 7 & 11 & 10 \\ 4 & 8 & -4 & 12 & 16 & 16 \end{bmatrix} = X$$

$$\boxed{C = \begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 4 & 8 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 3 & 2 \end{bmatrix} \quad \text{and } X = CR}$$

f) When we transpose $A = CR$, we get $A^T = (CR)^T = R^T C^T$. This equation also expresses A^T as a product of two matrices. Explain why $A^T = R^T C^T$ is also a CR decomposition of A^T. (Verify that the conditions for a CR decomposition are satisfied.)

Solution:
We start with the CR decomposition of A:

$$A = CR$$

where the columns of C are linearly independent and form a basis for colsp(A), and the rows of R are linearly independent and form a basis for colsp(A^T).

Taking the transpose of both sides gives

$$A^T = (CR)^T = R^T C^T$$

We will now verify that this is also a valid CR decomposition for A^T.

If A is an $n \times d$ matrix with rank r, then

$$C \in \mathbb{R}^{n\times r} \quad \text{and} \quad R \in \mathbb{R}^{r\times d}$$

After transposing, we have

$$R^T \in \mathbb{R}^{d\times r} \quad \text{and} \quad C^T \in \mathbb{R}^{r\times n}$$

and therefore

$$A^T = R^T C^T$$

has the correct shape: $(d \times r)(r \times n) = d \times n$.

The rows of R are linearly independent by the definition of the CR decomposition of A. Transposing R turns these rows into columns, which means the columns of R^T are linearly independent. Thus, R^T plays the same role for A^T that C played for A — its columns form a basis for colsp(A^T).

Similarly, the columns of C are linearly independent, so the rows of C^T are linearly independent. This means the rows of C^T form a basis for colsp$((A^T)^T) =$ colsp(A).

Since:

- the columns of R^T are linearly independent and form a basis for colsp(A^T), and
- the rows of C^T are linearly independent and form a basis for colsp$((A^T)^T)$,

all conditions for a CR decomposition are satisfied.
Therefore,

$$\boxed{A^T = R^T C^T \text{ is a valid CR decomposition of } A^T.}$$

For reference, see **Chapter 2.8: CR Decomposition**.

The key idea being assessed here is that in $A = CR$, the columns of C are linearly independent and a basis for colsp(A), while the rows of R are linearly independent and a basis for colsp(A^T).

Activity 4: Outer Products

Suppose $A = \vec{u}\vec{v}^T + \vec{w}\vec{z}^T$, where $\vec{u}, \vec{v}, \vec{w}, \vec{z} \in \mathbb{R}^n$ are non-zero vectors.

a) What is rank$(\vec{u}\vec{v}^T)$?

Solution:
We want to determine the rank of $\vec{u}\vec{v}^T$, where $\vec{u}, \vec{v} \in \mathbb{R}^n$ are nonzero vectors.

The matrix $\vec{u}\vec{v}^T$ is formed by multiplying a column vector $\vec{u}$ by a row vector $\vec{v}^T$:

$$\vec{u}\vec{v}^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & \cdots & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nv_1 & u_nv_2 & \cdots & u_nv_n \end{bmatrix}$$

Each column of this matrix is a scalar multiple of the same vector $\vec{u}$. For example:

First column: $v_1\vec{u}$, Second column: $v_2\vec{u}$, $\ldots$, n-th column: $v_n\vec{u}$.

Thus, every column lies in the same one-dimensional subspace spanned by $\vec{u}$:

$$\text{colsp}(\vec{u}\vec{v}^T) = \text{span}(\{\vec{u}\})$$

Since all columns are proportional to $\vec{u}$, there is exactly one linearly independent column. The dimension of the column space (and hence the rank) is therefore 1:

$$\boxed{\text{rank}(\vec{u}\vec{v}^T) = 1}$$

b) Find a basis for colsp(A). *Hint: What happens if you multiply A by a vector $\vec{x} \in \mathbb{R}^n$?*

Solution:
We are given

$$A = \vec{u}\vec{v}^T + \vec{w}\vec{z}^T$$

where $\vec{u}, \vec{v}, \vec{w}, \vec{z} \in \mathbb{R}^n$ are nonzero vectors, and we want to find a basis for colsp(A).

Consider what happens when A multiplies an arbitrary vector $\vec{x} \in \mathbb{R}^n$:

$$A\vec{x} = (\vec{u}\vec{v}^T + \vec{w}\vec{z}^T)\vec{x} = \vec{u}(\vec{v}^T\vec{x}) + \vec{w}(\vec{z}^T\vec{x})$$

This shows that $A\vec{x}$ is always a linear combination of $\vec{u}$ and $\vec{w}$. Thus,

$$\text{colsp}(A) \subseteq \text{span}(\{\vec{u}, \vec{w}\})$$

- If $\vec{u}$ and $\vec{w}$ are linearly independent, colsp$(A) = \text{span}(\{\vec{u}, \vec{w}\})$.
- If $\vec{u}$ and $\vec{w}$ are linearly dependent, colsp$(A) = \text{span}(\{\vec{u}\}) = \text{span}(\{\vec{w}\})$.

$$\boxed{\text{Basis for colsp}(A) = \begin{cases} \{\vec{u}, \vec{w}\}, & \text{if } \vec{u} \text{ and } \vec{w} \text{ are linearly independent,} \\ \{\vec{u}\}, & \text{if } \vec{u} \text{ and } \vec{w} \text{ are linearly dependent.} \end{cases}}$$

c) Find a basis for colsp(A^T).

Solution:
We have

$$A = \vec{u}\vec{v}^T + \vec{w}\vec{z}^T \quad \Longrightarrow \quad A^T = \vec{v}\vec{u}^T + \vec{z}\vec{w}^T$$

This has the same structure as A, but with the roles of $(\vec{u}, \vec{w})$ and $(\vec{v}, \vec{z})$ swapped. From part (b), we know that the column space of a matrix of this form is spanned by the first vector in each outer product term:

$$\text{colsp}(A^T) \subseteq \text{span}(\{\vec{v}, \vec{z}\})$$

- If $\vec{v}$ and $\vec{z}$ are linearly independent, colsp$(A^T) = \text{span}(\{\vec{v}, \vec{z}\})$.
- If $\vec{v}$ and $\vec{z}$ are linearly dependent, colsp$(A^T) = \text{span}(\{\vec{v}\}) = \text{span}(\{\vec{z}\})$.

$$\boxed{\text{Basis for colsp}(A^T) = \begin{cases} \{\vec{v}, \vec{z}\}, & \text{if } \vec{v} \text{ and } \vec{z} \text{ are linearly independent,} \\ \{\vec{v}\}, & \text{if } \vec{v} \text{ and } \vec{z} \text{ are linearly dependent.} \end{cases}}$$

d) Under what conditions is rank$(A) < 2$?

Solution:
We are given
$$A = \vec{u}\vec{v}^T + \vec{w}\vec{z}^T$$
with each outer product being rank 1. The sum will have rank 2 only if the two terms contribute independent directions in both the column and row spaces.

If either pair of vectors is linearly dependent, the total rank drops to 1:

- If $\vec{u}$ and $\vec{w}$ are linearly dependent, both terms point in the same column direction, so $\text{colsp}(A) = \text{span}(\{\vec{u}\})$.
- If $\vec{v}$ and $\vec{z}$ are linearly dependent, both terms share the same row direction, so $\text{rowsp}(A) = \text{span}(\{\vec{v}\})$.

Either case implies $\text{rank}(A) = 1$. If one or more of the vectors is $\vec{0}$, the rank is also less than 2.

$$\boxed{\text{rank}(A) < 2 \text{ if and only if } (\vec{u}, \vec{w}) \text{ are linearly dependent or } (\vec{v}, \vec{z}) \text{ are linearly dependent.}}$$

Activity 5: The Systems of Equations View

Let A be an $n \times d$ matrix of rank r, and suppose there exists vectors $\vec{b} \in \mathbb{R}^n$ such that

$$A\vec{x} = \vec{b}$$

does not have a solution (meaning no $\vec{x}$ makes $A\vec{x} = \vec{b}$).

a) What are all inequalities ($<$ or $\leq$) that must be true between n, d, and r?

Solution:
If $A\vec{x} = \vec{b}$ has no solution for some $\vec{b}$, then $\vec{b}$ does not lie in colsp(A). Thus, the column space of A is a proper subspace of $\mathbb{R}^n$, meaning its dimension (the rank) is strictly less than n:

$$r < n$$

By general rank properties, rank cannot exceed either the number of rows or columns:

$$r \leq n, \quad r \leq d$$

Combining these gives:

$$\boxed{r < n \quad \text{and} \quad r \leq d}$$

b) How do you know that $A^T\vec{y} = \vec{0}$ has solutions other than $\vec{y} = \vec{0}$?

Solution:
By the Rank–Nullity Theorem,

$$\text{rank}(A^T) + \dim(\text{nullsp}(A^T)) = n$$

and since rank$(A^T) = r < n$, it follows that $\dim(\text{nullsp}(A^T)) = n - r > 0$. Therefore, nullsp$(A^T)$ contains nonzero vectors:

$$\boxed{\exists\, \vec{y} \neq \vec{0} \text{ such that } A^T\vec{y} = \vec{0} \quad (\textit{the symbol } \exists \textit{ means "there exists"})}$$

We'll focus more on this systems of equations perspective in tomorrow's lecture, and this activity is an important warmup for the material.

Activity 6: Optional Practice

The rest of this worksheet is extra practice; don't feel compelled to answer all of these problems in lab, but make sure to attempt them at some point.

a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find a condition on a, b, c, d that ensures $\text{rank}(A) = 2$.

Solution:
We are given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and want a condition on a, b, c, d that guarantees $\text{rank}(A) = 2$.
The columns of A are

$$\vec{a}_1 = \begin{bmatrix} a \\ c \end{bmatrix} \qquad \vec{a}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$$

The rank of A equals 2 exactly when these two columns are linearly independent — that is, when neither column is a scalar multiple of the other.
If the columns were dependent, we would have

$$\frac{a}{b} = \frac{c}{d} \quad \text{(assuming the denominators are nonzero).}$$

Cross-multiplying gives $ad = bc$. Thus, for the columns to be independent, this equality must *not* hold:

$$ad - bc \neq 0$$

The quantity $ad - bc$ is called the **determinant** of A. When it is nonzero, the two columns (and rows) of A are linearly independent.

$$\boxed{\text{rank}(A) = 2 \quad \text{if and only if} \quad ad - bc \neq 0}$$

b) Find a matrix A such that $\begin{bmatrix} 3 \\ 1 \end{bmatrix} \in \text{colsp}(A)$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix} \in \text{nullsp}(A)$.

Solution:
We seek a 2×2 matrix A satisfying

$$A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix} \in \text{colsp}(A)$$

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The first condition gives

$$A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a + 3b \\ c + 3d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow a = -3b \quad c = -3d$$

Hence

$$A = \begin{bmatrix} -3b & b \\ -3d & d \end{bmatrix}$$

The columns of A are multiples of $\begin{bmatrix} -3 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so any vector in colsp$(A)$ has the form

$$A\vec{x} = \begin{bmatrix} b(-3x_1 + x_2) \\ d(-3x_1 + x_2) \end{bmatrix}$$

For this to equal $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, the ratios must match:

$$\frac{b}{d} = \frac{3}{1} \quad \Longrightarrow \quad b = 3d$$

Choosing $d = 1$ gives $b = 3$, $a = -9$, and $c = -3$:

$$A = \begin{bmatrix} -9 & 3 \\ -3 & 1 \end{bmatrix}$$

Quick check:

$$A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} -9 & 3 \\ -3 & 1 \end{bmatrix} \text{ satisfies both conditions.}}$$