Lab 6: Rank, Column Space, and Null Space

EECS 245, Fall 2025 at the University of Michigan

due by the end of your lab section on Wednesday, October 8th, 2025

Name: ______________________________

uniqname: ______________________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

This worksheet involves reinforcing ideas from **Chapter 2.8**. To briefly review, suppose A is an $n \times d$ matrix. Recall that rank(A) is the number of linearly independent columns of A.

	Notation	**Description**	**Dimension**	**Subspace of**
Column space	colsp(A)	Span of the columns of A	rank(A)	$\mathbb{R}^n$
Row space	colsp(A^T)	Span of the rows of A	rank(A)	$\mathbb{R}^d$
Nullspace	nullsp(A)	Set of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$	$d -$ rank(A)	$\mathbb{R}^d$

Activity 1: Programming (Do this at the end of the lab, not the beginning!)

Complete the tasks in the `lab06.ipynb` notebook, which you can either access through the DataHub link on the course homepage or by pulling our GitHub repository. To receive credit for Activity 1, show your lab TA that you've passed all test cases in your notebook; **there's no need to submit your notebook to Gradescope**.

Activity 2: Null Space of a Matrix with Linearly Independent Columns

Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \\ 1 & 0 \end{bmatrix}$. What is nullsp$(A)$?

Activity 3: Fundamentals

Let $X = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 & 4 \\ 2 & 5 & -2 & 7 & 11 & 10 \\ 4 & 8 & -4 & 12 & 16 & 16 \end{bmatrix}$.

a) Find a basis for $\text{colsp}(X)$. What is $\text{rank}(X)$? Why?

b) Fill in the blanks: $\text{colsp}(X^T)$ is a ____-dimensional subspace of ____.

c) Fill in the blanks: $\text{nullsp}(X)$ is a ____-dimensional subspace of ____.

d) Find a basis for $\text{nullsp}(X^T)$.

Suppose A is an $n \times d$ matrix with rank r. A CR decomposition of A is a product of two matrices C and R, where $A = CR$ and:

- C is an $n \times r$ matrix and R is a $r \times d$ matrix
- C contains the linearly independent columns of A, selected left-to-right
- R tells how to "mix" the columns of C (which are linearly independent) to reconstruct the columns of A

Let's keep working with $X = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 & 4 \\ 2 & 5 & -2 & 7 & 11 & 10 \\ 4 & 8 & -4 & 12 & 16 & 16 \end{bmatrix}$.

e) Find a CR decomposition of X. This shouldn't take very much work; review your work from part **a)** in finding a basis for $\text{colsp}(X)$.

f) When we transpose $A = CR$, we get $A^T = (CR)^T = R^T C^T$. This equation also expresses A^T as a product of two matrices. Explain why $A^T = R^T C^T$ is also a CR decomposition of A^T. (Verify that the conditions for a CR decomposition are satisfied.)

The key idea being assessed here is that in $A = CR$, the columns of C are linearly independent and a basis for $\text{colsp}(A)$, while the rows of R are linearly independent and a basis for $\text{colsp}(A^T)$.

Activity 4: Outer Products

Suppose $A = \vec{u}\vec{v}^T + \vec{w}\vec{z}^T$, where $\vec{u}, \vec{v}, \vec{w}, \vec{z} \in \mathbb{R}^n$ are non-zero vectors.

a) What is $\text{rank}(\vec{u}\vec{v}^T)$?

b) Find a basis for $\text{colsp}(A)$. *Hint: What happens if you multiply A by a vector $\vec{x} \in \mathbb{R}^n$?*

c) Find a basis for $\text{colsp}(A^T)$.

d) Under what conditions is $\text{rank}(A) < 2$?

Activity 5: The Systems of Equations View

Let A be an $n \times d$ matrix of rank r, and suppose there exists vectors $\vec{b} \in \mathbb{R}^n$ such that

$$A\vec{x} = \vec{b}$$

does not have a solution (meaning no $\vec{x}$ makes $A\vec{x} = \vec{b}$).

a) What are all inequalities ($<$ or $\leq$) that must be true between n, d, and r?

b) How do you know that $A^T\vec{y} = \vec{0}$ has solutions other than $\vec{y} = \vec{0}$?

We'll focus more on this systems of equations perspective in tomorrow's lecture, and this activity is an important warmup for the material.

Activity 6: Optional Practice

The rest of this worksheet is extra practice; don't feel compelled to answer all of these problems in lab, but make sure to attempt them at some point.

a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find a condition on a, b, c, d that ensures $\text{rank}(A) = 2$.

b) Find a matrix A such that $\begin{bmatrix} 3 \\ 1 \end{bmatrix} \in \text{colsp}(A)$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix} \in \text{nullsp}(A)$.