Lab 6: Vector Spaces, Subspaces, and Bases

EECS 245, Winter 2026 at the University of Michigan

due by the end of your lab section

Name: ______________________________

uniqname: ______________________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

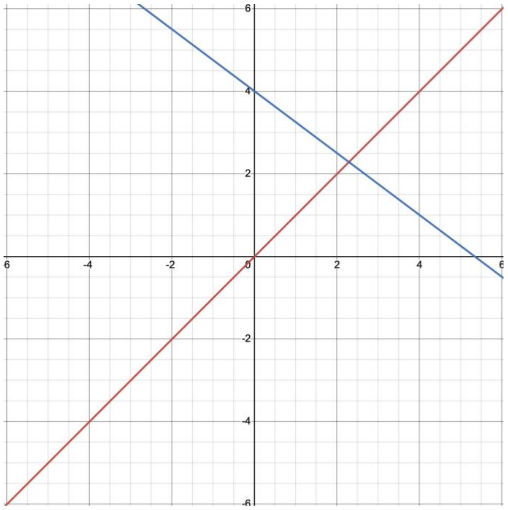
While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

Recap: Vector Spaces, Subspaces, and Bases (Chapter 4.3)

- A **subspace** S of a vector space V is a set of vectors where:
  1. $\vec{0} \in S$
  2. $\vec{u}, \vec{v} \in S \to \vec{u} + \vec{v} \in S$
  3. $\vec{u} \in S, c \in \mathbb{R} \to c\vec{u} \in S$

  If you take any two vectors $\vec{u}, \vec{v} \in S$, then any linear combination $c\vec{u} + d\vec{v}$ must also be in S.

- As an example, let's consider $\mathbb{R}^2$, which itself is a vector space.



  - The line through the origin **is** a subspace of $\mathbb{R}^2$, with dimension 1. It is the span of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
  - The other line, however, is **not** a subspace of $\mathbb{R}^2$, since it doesn't pass through the origin.

- A **basis** for a subspace S is a set of vectors that:
  1. span all of S
  2. are linearly independent

  A basis for a subspace is a minimal set of vectors that spans the whole subspace. All subspaces have infinitely many bases. For example, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ are both bases for $\mathbb{R}^2$.

- The **dimension** of a subspace S, denoted dim(S), is the number of vectors in any basis for S.

Activity 1: Introduction to Subspaces

Only one of the following is a subspace of $\mathbb{R}^3$. Which one? Explain why the others are not subspaces.

The set of vectors $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ such that

(i) $x + 2y - 3z = 4$

(ii) $\vec{v}$ is on the line $L = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, t \in \mathbb{R}$

(iii) $x + y + z = 0$ and $x - y + z = 1$

(iv) $x = -z$ and $x = z$

(v) $x^2 + y^2 = z$

Activity 2: Finding Non-Examples of Subspaces

In this activity, you'll find sets of vectors in $\mathbb{R}^2$ that satisfy some, but not all, of the requirements for a subspace. Think creatively, and since we're working in $\mathbb{R}^2$, visualize the vectors!

a) Find a set of vectors in $\mathbb{R}^2$ such that the sum of any two vectors $\vec{u}$ and $\vec{v}$ in the set is also in the set, but $\frac{1}{2}\vec{v}$ is possibly not in the set.

b) Find a set of vectors in $\mathbb{R}^2$ such that $c\vec{v}$ is in the set for any vector $\vec{v}$ in the set and any scalar c, but the sum of any two vectors $\vec{u}$ and $\vec{v}$ in the set is possibly not in the set.

Activity 3: Finding Bases for Subspaces

In each part below, find **two different possible bases** for the given subspace, and state the **dimension** of the subspace. (Note that this is effectively what you did in **Problem 4 of Homework 4** and **Activity 4 of Lab 5**, we just hadn't introduced the term "basis" at that point.)

a) $S = \text{span}\left(\left\{\begin{bmatrix}1\\3\\3\end{bmatrix}, \begin{bmatrix}-3\\-9\\-9\end{bmatrix}, \begin{bmatrix}1\\5\\-1\end{bmatrix}, \begin{bmatrix}2\\7\\4\end{bmatrix}, \begin{bmatrix}1\\4\\1\end{bmatrix}\right\}\right)$

b) $S = \left\{\begin{bmatrix}v_1\\v_2\end{bmatrix} \mid v_1 = -v_2; v_1, v_2 \in \mathbb{R}\right\}$

c) $S = \left\{\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} \mid v_4 = 0; v_1, v_2, v_3 \in \mathbb{R}\right\}$

Activity 4: Formal Definition of Linear Independence

Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d \in \mathbb{R}^n$, and that $\vec{b} \in \text{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\})$.

a) Give a one sentence English explanation of what it means for $\vec{b} \in \text{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\})$.

b) Suppose that $a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d = \vec{b}$ **and** $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_d\vec{v}_d = \vec{b}$, where at least one of the a_i's is different from its corresponding c_i.
Using the formal definition of linear independence from Chapter 4.2, determine whether or not $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are linearly independent, and prove your answer.

c) Find another set of coefficients $k_1, k_2, \ldots, k_d$ such that

$$k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots + k_d\vec{v}_d = \vec{b}$$

and at least one of the k_i's is different from its corresponding a_i or c_i.

By doing this, you're showing that if there is at least one way to write $\vec{b}$ as a linear combination of a set of vectors, then there are infinitely many ways to write $\vec{b}$ as a linear combination of those vectors; there can't just be two or three ways to do it.

Activity 5: More Linear Independence Practice

Let $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

a) Find scalars a, b, c, and d such that $a\vec{w} + b\vec{x} + c\vec{y} + d\vec{z} = \vec{0}$, and at least one of the scalars is non-zero. By doing so, you're showing that $\vec{w}, \vec{x}, \vec{y}, \vec{z}$ are linearly dependent.

b) Find scalars A, B, and C such that $\vec{z} = A\vec{w} + B\vec{x} + C\vec{y}$. This is another way of showing that $\vec{w}, \vec{x}, \vec{y}, \vec{z}$ are linearly dependent.

c) Show that $\text{span}(\{\vec{w}, \vec{x}, \vec{y}, \vec{z}\}) \neq \mathbb{R}^4$ by finding a vector $\vec{v} \in \mathbb{R}^4$ such that $\vec{v} \notin \text{span}\{\vec{w}, \vec{x}, \vec{y}, \vec{z}\}$.

d) Why is the fact that $\text{span}(\{\vec{w}, \vec{x}, \vec{y}, \vec{z}\}) \neq \mathbb{R}^4$ enough to conclude that $\vec{w}, \vec{x}, \vec{y}, \vec{z}$ are linearly **dependent**?