

Lab 7: Inverses **Solutions**

EECS 245, Fall 2025 at the University of Michigan

due by the end of your lab section on Wednesday, October 8th, 2025

Name: _____

username: _____

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

Activity 1: Basics of Invertibility

Suppose A is an $n \times n$ matrix. [Chapter 2.9](#) describes several equivalent conditions that guarantee that A is invertible. State as many of these equivalent conditions as you can, **without** looking at the notes.

Solution: If A is an $n \times n$ matrix, then A is invertible if and only if:

- $\text{rank}(A) = n$
- A 's columns are linearly independent (and hence $\text{colsp}(A) = \mathbb{R}^n$)
- A 's rows are linearly independent (and hence $\text{rowsp}(A) = \text{colsp}(A^T) = \mathbb{R}^n$)
- A 's null space is only the zero vector, i.e. $\text{nullsp}(A) = \{\vec{0}\}$
- $\det(A) \neq 0$

Activity 2: Symbolic Inverses

Given that A is an invertible $n \times n$ matrix that satisfies $A^4 - 3A^2 + 2A - 4I = 0$, find an expression for A^{-1} in terms of A .

Solution: The goal is to find another matrix B such that $AB = BA = I$. We can do this by isolating the identity matrix, I , and then trying to write the other side of the equation as A times some other matrix.

$$A^4 - 3A^2 + 2A - 4I = 0$$

$$A^4 - 3A^2 + 2A = 4I$$

$$A(A^3 - 3A + 2I) = 4I$$

$$A \left(\frac{1}{4}(A^3 - 3A + 2I) \right) = I$$

So, $A^{-1} = \frac{1}{4}(A^3 - 3A + 2I)$.

We derived A^{-1} by factoring out A on the left, i.e. $A \left(\frac{1}{4}(A^3 - 3A + 2I) \right) = I$. For A^{-1} to be the inverse of A , it must also be true that $\left(\frac{1}{4}(A^3 - 3A + 2I) \right) A = I$; this fact is not automatically true in general, since the order of multiplication matters. But here, we don't need to do any additional work, since our factorization only involved powers of A , and in (say) $A^2 = AA$ the order of multiplication doesn't matter.

Activity 3: True or False?

In each part, either prove that the statement is true or provide a counterexample.

- a) If A and B are both invertible $n \times n$ matrices, then $A + B$ is invertible.

Solution: False. Suppose $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. These are both invertible matrices, with $A^{-1} = A$ and $B^{-1} = B$.

But

$$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is not invertible, since its columns don't span $\mathbb{R}^2$.

- b) If A^2 is invertible, then A is invertible.

Solution: True. There are a few ways to reason about this, all of which are important.

Solution 1: By direct construction

One way to prove that A is invertible is to find its inverse. Since A^2 is invertible, there must exist some matrix B such that

$$A^2B = I, \quad BA^2 = I$$

Note that the inclusion of both orders is important, since in general, order matters for matrix multiplication. Writing $A^2 = AA$ in the first equation gives us

$$A(AB) = I$$

This tells us that AB is the inverse of A , so A is invertible.

If we used the second equation instead, we'd have

$$(BA)A = I$$

So, we'd call AB the right inverse of A (since it's the inverse that multiplies on the right) and BA the left inverse. A general fact is that if A has both a left and right inverse, they must be the same, meaning that AB must be equal to BA here (remember this is not true in general for arbitrary matrices A and B). A quick proof of this is

$$BA = BAI = BA(AAB) = BAAAB = (BAA)AB = IAB = AB$$

So, in short, $A^{-1} = AB = BA$, and since we've found A 's inverse, A must be invertible.

Solution 2: The null space perspective

Let's argue by contradiction. Suppose A^2 is invertible, but A is not. This would imply that A^2 null space is just the zero vector, $\vec{0}$, while A 's null space is not just $\vec{0}$. Suppose $\vec{x}$ is some vector **other than** $\vec{0}$ in $\text{nullsp}(A)$, meaning that

$$A\vec{x} = \vec{0}$$

But, left-multiplying both sides by A gives us

$$A(A\vec{x}) = A\vec{0} \implies A^2\vec{x} = \vec{0}$$

This tells us that $\vec{x}$ is in the null space of A^2 , since $A^2\vec{x} = \vec{0}$. But, this contradicts our assumption that A^2 is invertible, because the only vector in the null space of an invertible matrix is the zero vector. So, our original assumption that A is not invertible must be false, and A must be invertible.

Solution 3: The determinant perspective

A fact buried in [Chapter 2.9](#) is that if A and B are two $n \times n$ matrices, then

$$\det(A)\det(B) = \det(AB)$$

So,

$$\det(A^2) = \det(A \cdot A) = \det(A) \cdot \det(A) = \det(A)^2$$

But, we also know that invertible matrices have non-zero determinants, so we're given that $\det(A^2) \neq 0$. But, since $\det(A^2) = \det(A)^2$, this implies that $\det(A)^2 \neq 0$, and so $\det(A) \neq 0$ too, and so A is invertible.

c) If $A \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ 3 \end{bmatrix}$ and $A \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then A is invertible. (What could $\text{rank}(A)$ be?)

Solution: False. A is not invertible, because its null space contains more than just the zero vector.

$$\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \in \text{nullsp}(A)$$

We're also given some non-zero vector in the column space of A :

$$\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \in \text{colsp}(A)$$

A doesn't have rank 3, because then A would have a trivial null space of just $\vec{0}$, and it doesn't have rank 0, because then A would not have any vectors in its column space other than $\vec{0}$.

So, $\text{rank}(A) \in \{1, 2\}$.

Activity 4: The 2×2 Case

Recall that the inverse of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Using the fact above, find scalars x_1 and x_2 such that

$$2x_1 - 3x_2 = 6$$

$$5x_1 + 5x_2 = 10$$

Hint: First, write the system of equations in the form $A\vec{x} = \vec{b}$. If A is invertible, and $A\vec{x} = \vec{b}$, then what is $\vec{x}$?

Solution: Written in matrix-vector form, the system of equations is $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 2 & -3 \\ 5 & 5 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

Then, the scalars x_1 and x_2 that satisfy the system are contained in the vector $\vec{x}$, where

$$\vec{x} = A^{-1}\vec{b}$$

Using the provided formula for A^{-1} , we have

$$A^{-1} = \frac{1}{2 \cdot 5 - (-3) \cdot 5} \begin{bmatrix} 5 & -(-3) \\ -5 & 2 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 5 & 3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1/5 & 3/25 \\ -1/5 & 2/25 \end{bmatrix}$$

So,

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 1/5 & 3/25 \\ -1/5 & 2/25 \end{bmatrix} \begin{bmatrix} 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 60/25 \\ -10/25 \end{bmatrix} = \begin{bmatrix} 12/5 \\ -2/5 \end{bmatrix}$$

Therefore, $x_1 = 12/5$ and $x_2 = -2/5$.

Activity 5: Thinking in Transformations

Suppose $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear transformation represented by the matrix A .

Furthermore, suppose that $f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$, $f\left(\begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix}$, and $f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

a) Find $f\left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right)$. **After that**, find the matrix A corresponding to f , i.e. where $f(\vec{x}) = A\vec{x}$.

Solution: First, using just the properties of linear transformations, we can find $f\left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right)$. Recall that a linear transformation f satisfies

- $f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$
- $f(c\vec{x}) = cf(\vec{x})$

In other words, f preserves linear combinations, i.e. $f(a\vec{x} + b\vec{y}) = af(\vec{x}) + bf(\vec{y})$.

Let's decompose $f\left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right)$ into a linear combination of outputs we already know.

$$\begin{aligned} f\left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right) &= f\left(2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{10}\begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\ &= \underbrace{2f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + \frac{1}{10}f\left(\begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}\right) + 2f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)}_{\text{property of linear transformations}} \\ &= 2\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + \frac{1}{10}\begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 4/10 \\ -3/10 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 6.4 \\ 7.7 \end{bmatrix} \end{aligned}$$

$$\text{So, } f\left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 6.4 \\ 7.7 \end{bmatrix}.$$

Next, we need to find the matrix A corresponding to f , i.e. where $f(\vec{x}) = A\vec{x}$. Since f is a linear transformation, we know that $f(\vec{x}) = A\vec{x}$ for some matrix A .

- The first column of A is given by $f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$, which we're told is $\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$.
- The second column of A is given by $f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$, which we've computed above to be $\begin{bmatrix} 0 \\ 4/10 \\ -3/10 \end{bmatrix}$.

- The third column of A is given by $f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$, which we're told is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

So,

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 4/10 & 0 \\ 4 & -3/10 & 0 \end{bmatrix}$$

If that seems too simple to be true, you can verify that multiplying A by $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ gives us

$\begin{bmatrix} 4 \\ 6.4 \\ 7.7 \end{bmatrix}$, and that multiplying A by the three vectors in the question give the outputs we're

told. Remember that $A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ returns just the first column of A , and so on.

- b) Find a **diagonal** matrix D and an **orthogonal** matrix Q such that $A = QD$. (Not every matrix can be written in this form, but this particular A can.) Then, describe **in English** how f transforms a vector $\vec{x}$.

Solution:

Remember that

- Diagonal matrices have 0s everywhere except on the diagonal, and have the effect of stretching/compressing each axis/dimension of the input vector independently.
- Orthogonal matrices have columns that are orthonormal, meaning their columns are unit vectors that are orthogonal to one another.

In

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 4/10 & 0 \\ 4 & -3/10 & 0 \end{bmatrix}$$

You might notice that A 's first column is 5 times $\begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix}$, which is a unit vector. You

might also notice that A 's second column is $(1/2)$ times $\begin{bmatrix} 0 \\ 4/5 \\ -3/5 \end{bmatrix}$, which is another unit vector orthogonal to the first column. And finally, A 's third column is already a unit vector, and its orthogonal to the first two columns.

So, if we put these orthogonal unit vectors into the columns of Q , and the scaling factors of 5, $1/2$, and 1 into the diagonal of D , we have

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 4/10 & 0 \\ 4 & -3/10 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 3/5 & 4/5 & 0 \\ 4/5 & -3/5 & 0 \end{bmatrix}}_Q \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_D$$

$$\text{So, } Q = \begin{bmatrix} 0 & 0 & 1 \\ 3/5 & 4/5 & 0 \\ 4/5 & -3/5 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$f(\vec{x}) = QD\vec{x}$ transforms f by first scaling $\vec{x}$'s first component by 5, second component by $1/2$, and leaving the third component as-is, and then rotating it by the orthogonal matrix Q .

Orthogonal matrices rotate the vectors that they're multiplied by. In $\mathbb{R}^2$, this is a rotation by some angle θ . It's harder to describe rotations in $\mathbb{R}^3$ and beyond, but the key property that describes the effect of an orthogonal matrix Q is that for any vector $\vec{x}$,

$$\|Q\vec{x}\| = \|\vec{x}\|$$

as you proved in Homework 5, meaning that all an orthogonal matrix is doing is changing the angle of the vector, not its length.

So, $f(\vec{x})$ first scales $\vec{x}$'s components, and then rotates the resulting vector.

- c) Using your $A = QD$ decomposition from part b), find A^{-1} .

Hint: Recall that for orthogonal matrices, $QQ^T = Q^TQ = I$. And, for any invertible matrices A and B , $(AB)^{-1} = B^{-1}A^{-1}$.

Solution:

Since $A = QD$, we have

$$A^{-1} = D^{-1}Q^{-1}$$

Since D is diagonal, its inverse is just the diagonal matrix with the reciprocal of each diagonal entry. This corresponds to “unstretching” the vector in each dimension. So,

$$D^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And, since Q is orthogonal, we know $Q^TQ = I$, meaning that $Q^{-1} = Q^T$, i.e. Q ’s inverse is its transpose. This corresponds to “undoing” the rotation of the vector.

$$Q^{-1} = Q^T = \begin{bmatrix} 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \\ 1 & 0 & 0 \end{bmatrix}$$

Putting these building blocks together gives us

$$A^{-1} = D^{-1}Q^{-1} = D^{-1}Q^T = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3/25 & 4/25 \\ 0 & 8/5 & -6/5 \\ 1 & 0 & 0 \end{bmatrix}$$

- d) Recall from [Chapter 2.9](#) that the **determinant** of an $n \times n$ matrix A , $\det(A)$, describes how much the matrix scales the “volume” of an n -dimensional cube with side length 1.

Given the English definition of f from part b) **alone**, find $\det(A)$. (Don’t skip to the next page!)

Solution: Recall, $f(\vec{x})$ first scales $\vec{x}$ ’s first component by 5, second component by $1/2$, and leaves the third component as-is, and then it rotates the resulting vector in a way that preserves its length.

Intuitively, $\det(A)$ should be the product of the scaling factors, i.e. $5 \cdot 1/2 \cdot 1 = 5/2$.

e) In general, the determinant of a 3×3 matrix $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is given by

$$\underbrace{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}_{\det(M)} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

For instance, the $-b \begin{vmatrix} d & f \\ g & i \end{vmatrix}$ term in the determinant involves deleting row 1 and column 2 of M and taking the determinant of the remaining 2×2 matrix.

Use this formula directly on A from part **a)** to verify that your intuitive answer from part **d)** is correct.

Solution:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 4/10 & 0 \\ 4 & -3/10 & 0 \end{bmatrix}$$

Using the provided formula gives us

$$\begin{aligned} \det(A) &= 0 \cdot \begin{vmatrix} 4/10 & 0 \\ -3/10 & 0 \end{vmatrix} - 0 \cdot \begin{vmatrix} 3 & 0 \\ 4 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & 4/10 \\ 4 & -3/10 \end{vmatrix} \\ &= 0 + 0 + 1 \cdot (3 \cdot (-3/10) - 4/10 \cdot 4) \\ &= -\frac{9}{10} - \frac{16}{10} \\ &= -\frac{5}{2} \end{aligned}$$

Note that the sign of the determinant is negative; it's the absolute value of the determinant that tells us the stretching factor for the volume. The sign is a function of the orientation of the transformed parallelepiped (which is not particularly important for our purposes).

f) Find the determinant of

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

What do you notice?

Solution:

$$\begin{aligned}\det(B) &= 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= 1(45 - 48) - 2(36 - 42) + 3(32 - 35) \\ &= -3 + 12 - 9 \\ &= 0\end{aligned}$$

Since $\det(B) = 0$, B 's columns are linearly dependent, and so B is not invertible. We've seen this example B before;

$$\text{column } 3 = -\text{column } 1 + 2 \cdot \text{column } 2$$