# Lab 7: Inverses

EECS 245, Fall 2025 at the University of Michigan

**due** by the end of your lab section on Wednesday, October 8th, 2025

**Name:** ______________________________

**uniqname:** ______________________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

## Activity 1: Basics of Invertibility

Suppose $A$ is an $n \times n$ matrix. Chapter 2.9 describes several equivalent conditions that guarantee that $A$ is invertible. State as many of these equivalent conditions as you can, **without** looking at the notes.

## Activity 2: Symbolic Inverses

Given that $A$ is an invertible $n \times n$ matrix that satisfies $A^4 - 3A^2 + 2A - 4I = 0$, find an expression for $A^{-1}$ in terms of $A$.

## Activity 3: True or False?

In each part, either prove that the statement is true or provide a counterexample.

**a)** If $A$ and $B$ are both invertible $n \times n$ matrices, then $A + B$ is invertible.

**b)** If $A^2$ is invertible, then $A$ is invertible.

**c)** If $A \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ 3 \end{bmatrix}$ and $A \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then $A$ is invertible. (What could $\text{rank}(A)$ be?)

**Activity 4: The $2 \times 2$ Case**

Recall that the inverse of the $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Using the fact above, find scalars $x_1$ and $x_2$ such that

$$\begin{aligned} 2x_1 - 3x_2 &= 6 \\ 5x_1 + 5x_2 &= 10 \end{aligned}$$

*Hint: First, write the system of equations in the form $A\vec{x} = \vec{b}$. If $A$ is invertible, and $A\vec{x} = \vec{b}$, then what is $\vec{x}$?*

### Activity 5: Thinking in Transformations

Suppose $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation represented by the matrix $A$.

Furthermore, suppose that $f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\3\\4\end{bmatrix}$, $f\left(\begin{bmatrix}0\\10\\0\end{bmatrix}\right) = \begin{bmatrix}0\\4\\-3\end{bmatrix}$, and $f\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}$.

**a)** Find $f\left(\begin{bmatrix}2\\1\\2\end{bmatrix}\right)$. **After that**, find the matrix $A$ corresponding to $f$, i.e. where $f(\vec{x}) = A\vec{x}$.

**b)** Find a **diagonal** matrix $D$ and an **orthogonal** matrix $Q$ such that $A = QD$. (Not every matrix can be written in this form, but this particular $A$ can.) Then, describe **in English** how $f$ transforms a vector $\vec{x}$.

**c)** Using your $A = QD$ decomposition from part **b)**, find $A^{-1}$.

*Hint: Recall that for orthogonal matrices, $QQ^T = Q^TQ = I$. And, for any invertible matrices $A$ and $B$, $(AB)^{-1} = B^{-1}A^{-1}$.*

**d)** Recall from Chapter 2.9 that the **determinant** of an $n \times n$ matrix $A$, $\det(A)$, describes how much the matrix scales the "volume" of an $n$-dimensional cube with side length 1.

Given the English definition of $f$ from part **b) alone**, find $\det(A)$. (Don't skip to the next page!)

**e)** In general, the determinant of a $3 \times 3$ matrix $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is given by

$$\underbrace{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}_{\det(M)} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

For instance, the $-b \begin{vmatrix} d & f \\ g & i \end{vmatrix}$ term in the determinant involves deleting row 1 and column 2 of $M$ and taking the determinant of the remaining $2 \times 2$ matrix.

Use this formula directly on $A$ from part **a)** to verify that your intuitive answer from part **d)** is correct.

**f)** Find the determinant of

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

What do you notice?