Lab 11: Diagonalization

EECS 245, Fall 2025 at the University of Michigan

due by the end of your lab section on Wednesday, November 19th, 2025

Name: ________________________________

uniqname: ________________________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

Recap: Eigenvalues and Eigenvectors

Let $A = \begin{bmatrix} 6 & 3 \\ 3 & -2 \end{bmatrix}$.

- An **eigenvector** of A is a non-zero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$ for some scalar λ. The scalar λ is called the **eigenvalue** corresponding to $\vec{v}$. For A's eigenvectors, multiplying by A is equivalent to multiplying by a scalar.
- The **characteristic polynomial** of A is given by $p(\lambda) = \det(A - \lambda I)$.

  $$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 6-\lambda & 3 \\ 3 & -2-\lambda \end{vmatrix} = (6-\lambda)(-2-\lambda) - 3 \cdot 3 = \lambda^2 - 4\lambda - 21 = (\lambda+3)(\lambda-7)$$

- The eigenvalues of A are the roots of the characteristic polynomial, so $\lambda_1 = -3$ and $\lambda_2 = 7$.
  - The eigenvector $\vec{v}_1$ satisfies $A\vec{v}_1 = -3\vec{v}_1$.

    $$\begin{bmatrix} 6 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -3 \begin{bmatrix} a \\ b \end{bmatrix} \implies b = -3a$$

    So any vector of the form $\begin{bmatrix} a \\ -3a \end{bmatrix}$ ($a \neq 0$) is an eigenvector of A corresponding to the eigenvalue -3. We could pick $\boxed{\vec{v}_1 = \begin{bmatrix} 2 \\ -6 \end{bmatrix}}$.
  - The eigenvector $\vec{v}_2$ satisfies $A\vec{v}_2 = 7\vec{v}_2$. Another way to find it is to solve for the null space of $A - 7I = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}$. One vector in $\text{nullsp}(A - 7I)$ is $\boxed{\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$.

Recap: Diagonalization

- Since A has two linearly independent eigenvectors, it is **diagonalizable**, meaning we can write

$$A = V\Lambda V^{-1} = \underbrace{\begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 0.05 & -0.15 \\ 0.3 & 0.1 \end{bmatrix}}_{V^{-1}}$$

where V is an invertible matrix whose columns are the eigenvectors of A and Λ is a diagonal matrix with the eigenvalues of A on the diagonal.

  - The **algebraic multiplicity** of an eigenvalue, $\mathrm{AM}(\lambda_i)$, is the number of times it appears as a root of the characteristic polynomial.

$$p(\lambda) = (\lambda - \lambda_1)^{\mathrm{AM}(\lambda_1)} (\lambda - \lambda_2)^{\mathrm{AM}(\lambda_2)} \cdots (\lambda - \lambda_k)^{\mathrm{AM}(\lambda_k)}$$

  - The **geometric multiplicity** of an eigenvalue, $\mathrm{GM}(\lambda_i)$, is the dimension of the eigenspace corresponding to λ_i.

$$\mathrm{GM}(\lambda_i) = \dim(\mathrm{nullsp}(A - \lambda_i I)) = \#\text{ linearly independent eigenvectors corresponding to } \lambda_i$$

  - For any λ_i, $1 \le \mathrm{GM}(\lambda_i) \le \mathrm{AM}(\lambda_i)$.
  - A is diagonalizable if and only if $\mathrm{AM}(\lambda_i) = \mathrm{GM}(\lambda_i)$ for all eigenvalues λ_i. This ensures that A's eigenvectors form a basis of $\mathbb{R}^n$.

- $A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ has characteristic polynomial $p(\lambda) = (4-\lambda)^2$. The eigenvalue $\lambda = 4$ has algebraic multiplicity 2, but the corresponding eigenspace is only 1-dimensional:

$$\mathrm{nullsp}(A - 4I) = \mathrm{nullsp}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

so, $\lambda = 4$ has geometric multiplicity 1. Since $\mathrm{AM}(4) \neq \mathrm{GM}(4)$, A is **not diagonalizable**.

- $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ has characteristic polynomial $p(\lambda) = (1-\lambda)^2(3-\lambda)$, so $\lambda_1 = 1$ has $\mathrm{AM}(\lambda_1) = 2$ and $\lambda_2 = 3$ has $\mathrm{AM}(\lambda_2) = 1$. The eigenspace for $\lambda_1 = 1$ is 2-dimensional,

$$\mathrm{nullsp}(A - 1I) = \mathrm{nullsp}\left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

so $\mathrm{GM}(\lambda_1) = 2$. Since $\mathrm{AM}(\lambda_i) = \mathrm{GM}(\lambda_i)$ for all eigenvalues λ_i, A is **diagonalizable**.

$$A = V\Lambda V^{-1} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 \end{bmatrix}}_{V^{-1}}$$

- Diagonalizability is not the same as invertibility!

Activity 1: Fundamentals

Let $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

a) Find the characteristic polynomial of A, and use it to find the eigenvalues of A and their algebraic multiplicities. *Tip: Add the formula for the determinant of a* 3×3 *matrix to your cheat sheet.*

b) Find a basis for the eigenspace corresponding to each eigenvalue.

c) This particular A is diagonalizable. Diagonalize A by finding a matrix V and a diagonal matrix Λ such that $A = V\Lambda V^{-1}$.

Activity 2: Rapid Fire

The goal of this activity is to practice spotting eigenvalues and characteristic polynomials quickly. Recall that the sum of the eigenvalues of a matrix is equal to the trace of the matrix (which is the sum of the diagonal entries), and the product of the eigenvalues is equal to the determinant of the matrix.

a) A 2×2 matrix A has $\text{trace}(A) = 5$ and $\det(A) = 6$. What are the eigenvalues of A?

b) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$. What are the eigenvalues of A? Is A diagonalizable?
Hint: Use the fact that A is an upper triangular matrix. Chapter 5.1 has some hints.

c) A non-invertible 2×2 matrix has an eigenvalue of 5. What is its characteristic polynomial?

d) A 5×5 matrix has an eigenvalue of 0 with geometric multiplicity 2. What is its rank?

e) A 3×3 matrix A has $\det(A) = 20$ and two unique **positive integer** eigenvalues, one with $\text{AM}(\lambda_1) = 2$ and another with $\text{AM}(\lambda_2) = 1$. What are all possible values of λ_1 and λ_2?

Activity 3: Adjacency Matrices

Suppose that a Wolverine moves between three classic Ann Arbor spots: the Diag, Zingerman's, and the Big House.

- From the Diag, it **always** walks to Zingerman's.
- From the Big House, it **always** walks to Zingerman's.
- From Zingerman's, it is **equally likely** to walk to the Diag or the Big House.

a) Draw a state diagram (like the one in Chapter 5.1, Part 2) for this Markov chain. Make sure to clearly label the edges with their transition probabilities.

b) Find the adjacency matrix A.

c) In the long run, what fraction of time does the Wolverine spend at each of the three spots? What does this have to do with the eigenvalues and eigenvectors of A?

Activity 4: Quadratic Forms Return

Open Desmos in 3D mode at **desmos.com/3d** and write $z = x^2 + 2bxy + 16y^2$. This should show you a 3D surface along with a slider for b. Drag the slider to see how the shape of the surface changes for different b's. You should notice that depending on the value of b, the surface may or may not have a global minimum. Let's explore!

a) z is a quadratic form, $f(\vec{x}) = \vec{x}^T A\vec{x}$, where $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and A is a symmetric matrix. Find A.

b) For a vector-to-scalar function $f : \mathbb{R}^n \to \mathbb{R}$, the **Hessian** of f, denoted $\nabla^2 f$, is the $n \times n$ matrix of second partial derivatives of f. Find $\nabla^2 f$ for $f(\vec{x}) = \vec{x}^T A\vec{x}$.

c) A symmetric matrix A is **positive semidefinite** (PSD) if $\vec{v}^T A\vec{v} \geq 0$ for all $\vec{v} \in \mathbb{R}^n$. In English, this says that A is positive semidefinite if the quadratic form $f(\vec{v}) = \vec{v}^T A\vec{v}$ is always non-negative for all $\vec{v} \in \mathbb{R}^n$. Two relevant facts:

- A differentiable vector-to-scalar function f is **convex** if its Hessian is PSD.
- A symmetric matrix A is PSD if and only if all of its eigenvalues are non-negative.

Using the facts above, find the range of values b for which f is convex, and verify your answer by dragging the slider on Desmos.

We don't expect you to finish all of the following activities in lab, but you should complete them afterwards for practice; they're in scope for the Final Exam.

Activity 5: Symmetric Matrices

Suppose A is a symmetric $n \times n$ matrix, meaning $A = A^T$. Symmetric matrices have a special property that we'll use more in Thursday's lecture: **the eigenvectors corresponding to different eigenvalues are orthogonal**. Below, we provide a proof of this fact.

1. Let $\vec{v}_i$ be an eigenvector of A with eigenvalue λ_i, so $A\vec{v}_i = \lambda_i \vec{v}_i$.
2. Let $\vec{v}_j$ be an eigenvector of A with eigenvalue λ_j, where $\lambda_i \neq \lambda_j$, so $A\vec{v}_j = \lambda_j \vec{v}_j$.
3. The dot product of $\vec{v}_i$ and $A\vec{v}_j$ is $\vec{v}_i \cdot (A\vec{v}_j) = \vec{v}_i \cdot (\lambda_j \vec{v}_j) = \lambda_j (\vec{v}_i \cdot \vec{v}_j)$.
4. But also, $\vec{v}_i \cdot (A\vec{v}_j) = \vec{v}_i^T A \vec{v}_j = \vec{v}_i^T A^T \vec{v}_j = (A\vec{v}_i)^T \vec{v}_j = \lambda_i \vec{v}_i^T \vec{v}_j = \lambda_i (\vec{v}_i \cdot \vec{v}_j)$.
5. The final expressions in both cases are equal, so $\lambda_j (\vec{v}_i \cdot \vec{v}_j) = \lambda_i (\vec{v}_i \cdot \vec{v}_j)$.
6. Equivalently, $(\lambda_j - \lambda_i)(\vec{v}_i \cdot \vec{v}_j) = 0$. But since $\lambda_i \neq \lambda_j$, we must have $\vec{v}_i \cdot \vec{v}_j = 0$.

a) In which line did we use the fact that A is symmetric? Select the line below, and then in that line, circle the specific part of the equation that uses the fact that $A = A^T$.

○ 1 ○ 2 ○ 3 ○ 4 ○ 5 ○ 6

b) The **spectral theorem** states that any symmetric $n \times n$ matrix A can be diagonalized by an orthogonal matrix Q through

$$A = Q \Lambda Q^T$$

Explain how this relates to the "regular" diagonalization $A = V \Lambda V^{-1}$.

Activity 6: Positive Semidefinite Matrices

As mentioned in Activity 4, a symmetric matrix A is **positive semidefinite** if $\vec{v}^T A\vec{v} \geq 0$ for all $\vec{v} \in \mathbb{R}^n$.

a) Prove that if A is positive semidefinite, then all of its eigenvalues are non-negative. *Hint: You can assume that all of A's eigenvalues are real numbers; this is true for all symmetric matrices. Start by considering $\vec{v}_i^T A\vec{v}_i$ for an eigenvector $\vec{v}_i$ of A with eigenvalue λ_i.*

b) Prove that if all of A's eigenvalues are non-negative, then A is positive semidefinite. *Hint: A is symmetric, which means it has a special version of the eigenvector decomposition, hinted in the previous activity. Consider $\vec{x}^T A\vec{x}$ for some arbitrary (not necessarily an eigenvector) vector $\vec{x}$, and then replace A with its special eigenvector decomposition.*

Activity 7: Diagonalization in Action

Let A be a 3×3 with:

- eigenvalue $\lambda_1 = 3$ with eigenvector $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.
- eigenvalue $\lambda_2 = -2$ with the 2-dimensional eigenspace: span $\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right)$.

a) Find A. Feel free to use `numpy` to find an inverse for you and verify your answer.

b) Let V be the matrix whose columns are the eigenvectors of A and Λ be the diagonal matrix with the eigenvalues of A on the diagonal. In terms of V and Λ, what is A^8?

c) Suppose $\vec{x} \in \mathbb{R}^3$ is some vector. As $k \to \infty$, what does $A^k\vec{x}$ approach? Explain your answer in English.