Lab 12: Adjacency Matrices and Diagonalization

EECS 245, Winter 2026 at the University of Michigan

due by the end of your lab section

Name: ____________________

uniqname: ____________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

Recap: Diagonalization (Chapter 9.4)

- Since A has two linearly independent eigenvectors, it is **diagonalizable**, meaning we can write

$$A = V\Lambda V^{-1} = \underbrace{\begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 0.05 & -0.15 \\ 0.3 & 0.1 \end{bmatrix}}_{V^{-1}}$$

  where V is an invertible matrix whose columns are the eigenvectors of A and Λ is a diagonal matrix with the eigenvalues of A on the diagonal.

  - The **algebraic multiplicity** of an eigenvalue, $\text{AM}(\lambda_i)$, is the number of times it appears as a root of the characteristic polynomial.

  $$p(\lambda) = (\lambda - \lambda_1)^{\text{AM}(\lambda_1)}(\lambda - \lambda_2)^{\text{AM}(\lambda_2)} \cdots (\lambda - \lambda_k)^{\text{AM}(\lambda_k)}$$

  - The **geometric multiplicity** of an eigenvalue, $\text{GM}(\lambda_i)$, is the dimension of the eigenspace corresponding to λ_i.

    $\text{GM}(\lambda_i) = \dim(\text{nullsp}(A - \lambda_i I)) = \#$ linearly independent eigenvectors corresponding to λ_i

  - For any λ_i, $1 \leq \text{GM}(\lambda_i) \leq \text{AM}(\lambda_i)$.
  - A is diagonalizable if and only if $\text{AM}(\lambda_i) = \text{GM}(\lambda_i)$ for all eigenvalues λ_i. This ensures that A's eigenvectors form a basis of $\mathbb{R}^n$.

- $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ has characteristic polynomial $p(\lambda) = (2 - \lambda)^2$. The eigenvalue $\lambda = 2$ has algebraic multiplicity 2, but the corresponding eigenspace is only 1-dimensional:

$$\text{nullsp}(A - 2I) = \text{nullsp}\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

so, $\lambda = 2$ has geometric multiplicity 1. Since $\text{AM}(2) \neq \text{GM}(2)$, A is **not diagonalizable**.

- $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ has characteristic polynomial $p(\lambda) = (1-\lambda)^2(3-\lambda)$, so $\lambda_1 = 1$ has $\text{AM}(\lambda_1) = 2$ and $\lambda_2 = 3$ has $\text{AM}(\lambda_2) = 1$. The eigenspace for $\lambda_1 = 1$ is 2-dimensional,

$$\text{nullsp}(A - 1I) = \text{nullsp}\left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

so $\text{GM}(\lambda_1) = 2$. Since $\text{AM}(\lambda_i) = \text{GM}(\lambda_i)$ for all eigenvalues λ_i, A is **diagonalizable**.

$$A = V\Lambda V^{-1} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 \end{bmatrix}}_{V^{-1}}$$

- Diagonalizability is not the same as invertibility!

Activity 1: Rapid Fire

The goal here is to answer the problems **quickly** without working out the details.

a) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$. What are the eigenvalues of A? Is A diagonalizable?

Hint: Use the fact that A is an upper triangular matrix. What is $\det(A - \lambda I)$*?*

b) A 5×5 matrix has an eigenvalue of 0 with geometric multiplicity 2. What is its rank?

Activity 2: Fundamentals

Let $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

a) Find the characteristic polynomial of A, and use it to find the eigenvalues of A and their algebraic multiplicities.

b) Find a basis for the eigenspace corresponding to each eigenvalue.

c) This particular A is diagonalizable. Diagonalize A by finding a matrix V and a diagonal matrix Λ such that $A = V\Lambda V^{-1}$.

Activity 3: Adjacency Matrices

Suppose that each night, a Wolverine moves between three classic Ann Arbor spots: the Diag, Zingerman's, and the Big House.

- From the Diag, $\frac{2}{3}$ of the time it stays at the Diag, $\frac{1}{6}$ of the time it walks to Zingerman's, and $\frac{1}{6}$ of the time it walks to the Big House.
- From the Big House, it **always** walks to Zingerman's.
- From Zingerman's, it is **equally likely** to walk to the Diag or the Big House.

a) Draw a state diagram for this Markov chain. Make sure to clearly label the edges with their transition probabilities. (For help, see **Chapter 9.3**.)

b) Find the adjacency matrix, A, of this Markov chain.

c) Show that the long-run distribution of the Wolverine's locations is $\begin{bmatrix} p(\text{Diag}) \\ p(\text{Big House}) \\ p(\text{Zingerman's}) \end{bmatrix} = \begin{bmatrix} 6/13 \\ 3/13 \\ 4/13 \end{bmatrix}$.

Hint: Do this by finding the eigenvector of A corresponding to the eigenvalue 1. Since there are infinitely many such eigenvectors, find the one that satisfies the constraint that the components sum to 1.

d) Open a new Jupyter Notebook or interactive Python session in the Terminal. In it, run:

```
import numpy as np
A = np.array(...) # Replace with the adjacency matrix you found in b)
eigvals, eigvecs = np.linalg.eig(A)
```

Now, if you run `eigvals`, you should see

```
array([ 1.        ,  0.36037961, -0.69371294])
```

If you run `eigvecs[:, 0]` to access the eigenvector corresponding to the first eigenvalue in the array above, you should see

```
array([0.76822128, 0.38411064, 0.51214752])
```

These is **not** the eigenvector you found in the previous part. Why not? What did it return, and what expression can you run in code to find the exact answer you found in the previous part?

e) Run both of the following commands:

```
np.linalg.matrix_power(A, 20) @ np.array([[1/3], [1/3], [1/3]])
np.linalg.matrix_power(A, 21) @ np.array([[1/3], [1/3], [1/3]])
```

What do you see? Why?

f) Now, consider the adjacency matrix

$$B = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 1 & 1 & 0 \end{bmatrix}$$

Using `numpy`, find the eigenvalues of B. Then, run all of the following commands:

```
np.linalg.matrix_power(B, 20) @ np.array([[1/3], [1/3], [1/3]])
np.linalg.matrix_power(B, 21) @ np.array([[1/3], [1/3], [1/3]])
np.linalg.matrix_power(B, 22) @ np.array([[1/3], [1/3], [1/3]])
np.linalg.matrix_power(B, 23) @ np.array([[1/3], [1/3], [1/3]])
```

This Markov chain appears not to converge. **Why not?** Relate your answer to the discussion of “the dominant eigenvalue” in Chapter 9.3.

Activity 4: Symmetric Matrices

Suppose A is a symmetric $n \times n$ matrix, meaning $A = A^T$. Symmetric matrices have a special property, described by the spectral theorem: **the eigenvectors corresponding to different eigenvalues are orthogonal**. Below, we provide a proof of this fact.

1. Let $\vec{v}_i$ be an eigenvector of A with eigenvalue λ_i, so $A\vec{v}_i = \lambda_i \vec{v}_i$.
2. Let $\vec{v}_j$ be an eigenvector of A with eigenvalue λ_j, where $\lambda_i \neq \lambda_j$, so $A\vec{v}_j = \lambda_j \vec{v}_j$.
3. The dot product of $\vec{v}_i$ and $A\vec{v}_j$ is $\vec{v}_i \cdot (A\vec{v}_j) = \vec{v}_i \cdot (\lambda_j \vec{v}_j) = \lambda_j(\vec{v}_i \cdot \vec{v}_j)$.
4. But also, $\vec{v}_i \cdot (A\vec{v}_j) = \vec{v}_i^T A \vec{v}_j = \vec{v}_i^T A^T \vec{v}_j = (A\vec{v}_i)^T \vec{v}_j = \lambda_i \vec{v}_i^T \vec{v}_j = \lambda_i(\vec{v}_i \cdot \vec{v}_j)$.
5. The final expressions in both cases are equal, so $\lambda_j(\vec{v}_i \cdot \vec{v}_j) = \lambda_i(\vec{v}_i \cdot \vec{v}_j)$.
6. Equivalently, $(\lambda_j - \lambda_i)(\vec{v}_i \cdot \vec{v}_j) = 0$. But since $\lambda_i \neq \lambda_j$, we must have $\vec{v}_i \cdot \vec{v}_j = 0$.

a) In which line did we use the fact that A is symmetric? Select the line below, and then in that line, circle the specific part of the equation that uses the fact that $A = A^T$.

◯ 1 ◯ 2 ◯ 3 ◯ 4 ◯ 5 ◯ 6

b) The **spectral theorem** states that any symmetric $n \times n$ matrix A can be diagonalized by an orthogonal matrix Q through

$$A = Q\Lambda Q^T$$

Explain how this relates to the "regular" diagonalization $A = V\Lambda V^{-1}$. Why does the equation above contain Q^T instead of Q^{-1}?

c) Symmetric matrices play a big role in solving optimization problems in machine learning. You'll get a taste of this in Homework 10, Problem 6, which discusses *ridge regression* and *regularization*, ideas that we use to make sure that our models are not overfitting to training data.

Here, we'll prove a related fact: **if A is an $n \times n$ symmetric matrix and all of its eigenvalues are non-negative, then A is positive semidefinite.** A symmetric matrix A is positive semidefinite if and only if $\boxed{\vec{v}^T A \vec{v} \geq 0}$ for all $\vec{v} \in \mathbb{R}^n$. One interpretation: this means the quadratic form $f(\vec{v}) = \vec{v}^T A \vec{v}$ is always greater than or equal to 0, no matter what $\vec{v}$ is.

Let's work through the proof step-by-step: we will do some of it for you, and you'll complete the rest. (To be precise, we're only proving one direction of the "if and only if" statement; the other direction is in Homework 10!) First, since A is symmetric, **the spectral theorem tells us that A can be written as**

$$A = Q\Lambda Q^T$$

where Q is an orthogonal matrix and Λ is a diagonal matrix with the eigenvalues of A on the diagonal. This is the eigenvector decomposition of A.

On top of that, suppose that all of A's eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, are non-negative.

Now, let $\vec{v}$ be some arbitrary vector (not necessarily an eigenvector of A) in $\mathbb{R}^n$. Then, eventually we need to show that $\vec{v}^T A \vec{v} \geq 0$, regardless of what $\vec{v}$ is. Let's start by expanding $\vec{v}^T A \vec{v}$ using the fact that $A = Q\Lambda Q^T$.

$$\vec{v}^T A \vec{v} = \vec{v}^T (Q\Lambda Q^T)\vec{v} = (\vec{v}^T Q)\Lambda(Q^T \vec{v})$$

Suppose that $\vec{y} = Q^T \vec{v}$. Then, $\vec{y}^T = (Q^T \vec{v})^T = \vec{v}^T Q$. This seems like an arbitrary maneuver, but it will be useful in a moment.

$$\vec{v}^T A \vec{v} = \vec{y}^T \Lambda \vec{y} = \begin{bmatrix} y_1 & y_2 & \ldots & y_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Your job is to complete the rest of the proof. Show that if A is an $n \times n$ symmetric matrix and all of its eigenvalues are non-negative, then $\vec{v}^T A \vec{v} \geq 0$ for all $\vec{v} \in \mathbb{R}^n$.

Activity 5: More Practice (Optional)

Let A be a 3×3 with:

- eigenvalue $\lambda_1 = 3$ with eigenvector $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.
- eigenvalue $\lambda_2 = -2$ with the 2-dimensional eigenspace: span $\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right)$.

a) Find A. Feel free to use `numpy` to find an inverse for you and verify your answer.

b) Let V be the matrix whose columns are the eigenvectors of A and Λ be the diagonal matrix with the eigenvalues of A on the diagonal. In terms of V and Λ, what is A^8?

c) Suppose $\vec{x} \in \mathbb{R}^3$ is some vector. As $k \to \infty$, what does $A^k\vec{x}$ approach? Explain your answer in English.