EECS 245, Winter 2026

LEC 6 The Dot Product

Read: Ch. 3

Agenda

3.1

- Recap: linear combinations
- Dot Product } 3.3
  - Definition
  - Properties
  - The "other" definition
  - Orthogonality
  - Inequalities (if time)

Read Ch. 3 this week!

Announcements

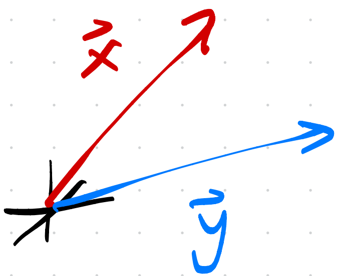
- HW 3 due Monday
- HW 4 released Monday, due following Monday
- HW 2 scores released
- Added more remote OH; still have OH after lecture

Example from last class

$\vec{x}, \vec{y} \in \mathbb{R}^3$

$\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ $\quad$ $\vec{y} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

Goal: write $\vec{b} = \begin{bmatrix} 9 \\ -16 \\ -1 \end{bmatrix}$ as a linear combination of $\vec{x}$ and $\vec{y}$.



What does that mean?

$$\vec{b} = c\vec{x} + d\vec{y}$$

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d \in \mathbb{R}^n$ "building blocks"

linear combination of them:

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3 + \cdots + a_d \vec{v}_d$$

$$-2\vec{v}_1 + 3\vec{v}_2 - 6\vec{v}_3 + \vec{v}_{17}$$

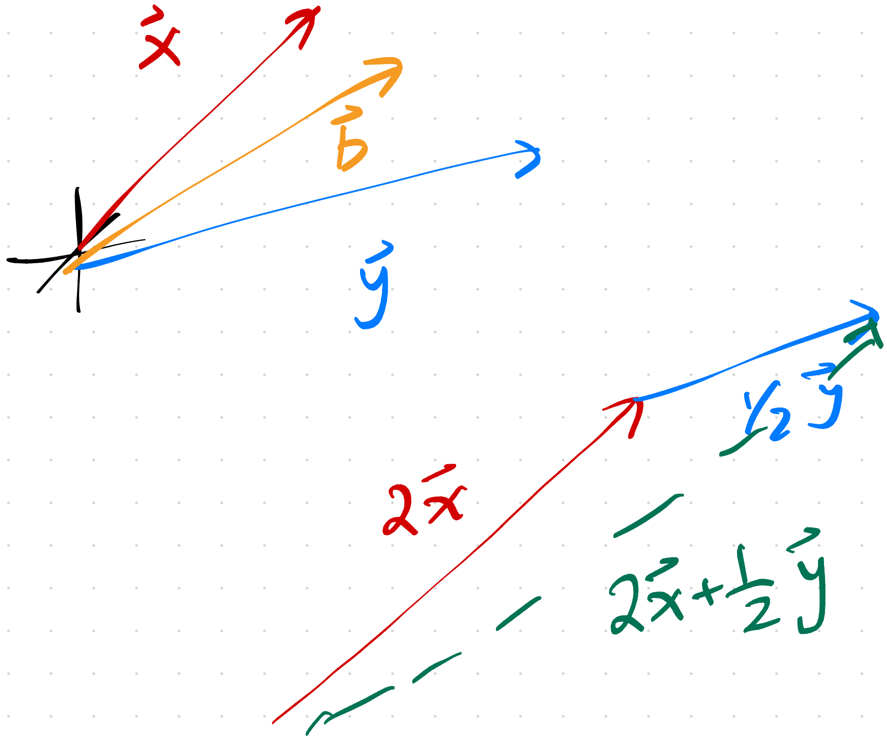
$$\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 9 \\ -16 \\ -1 \end{bmatrix}$$

$$c\vec{x} + d\vec{y} = \vec{b}$$

$$c\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + d\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ -16 \\ -1 \end{bmatrix}$$

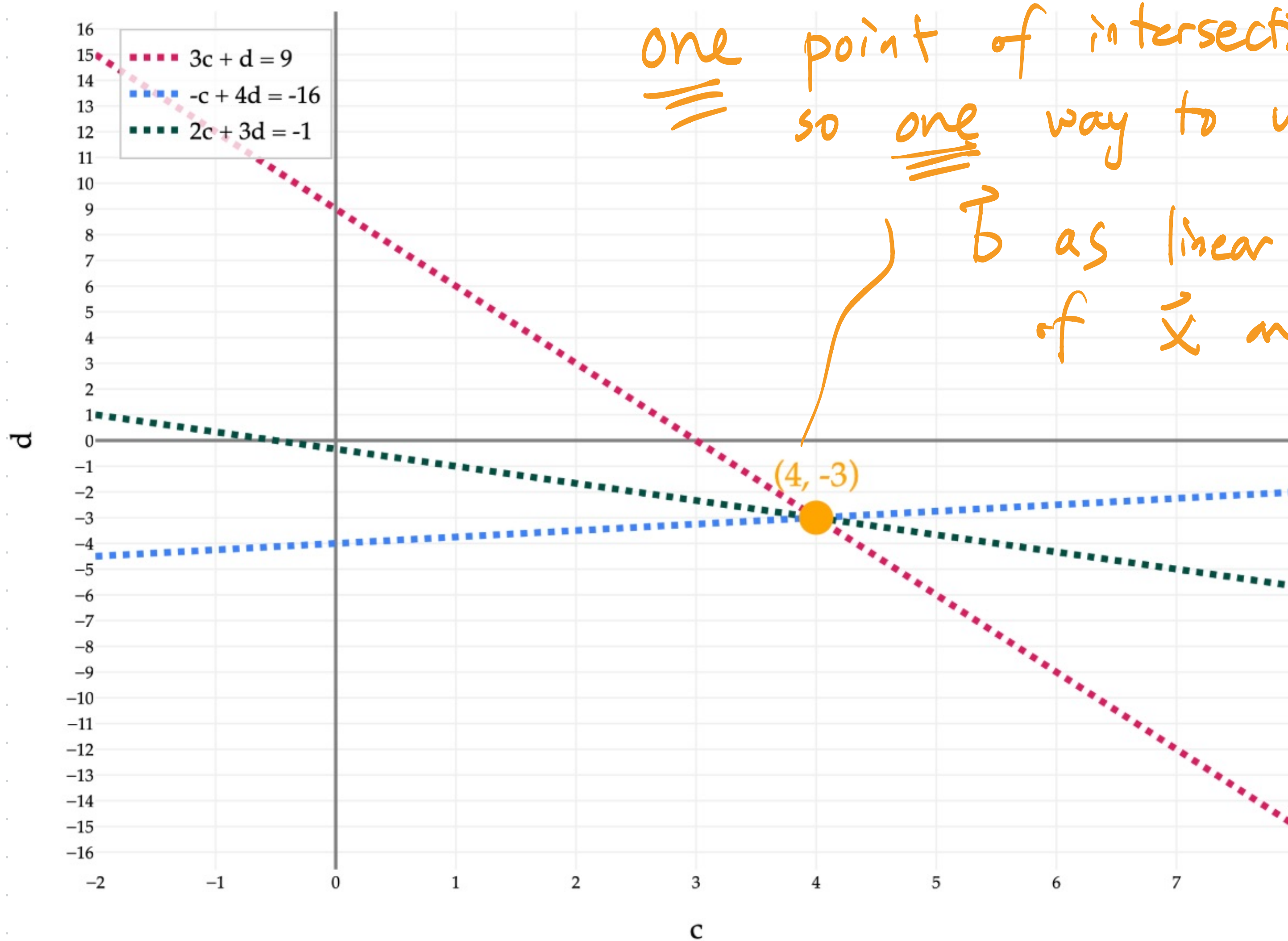
$$\begin{aligned} 3c + d &= 9 \\ -c + 4d &= -16 \\ 2c + 3d &= -1 \end{aligned}$$

→ solve for c and d!

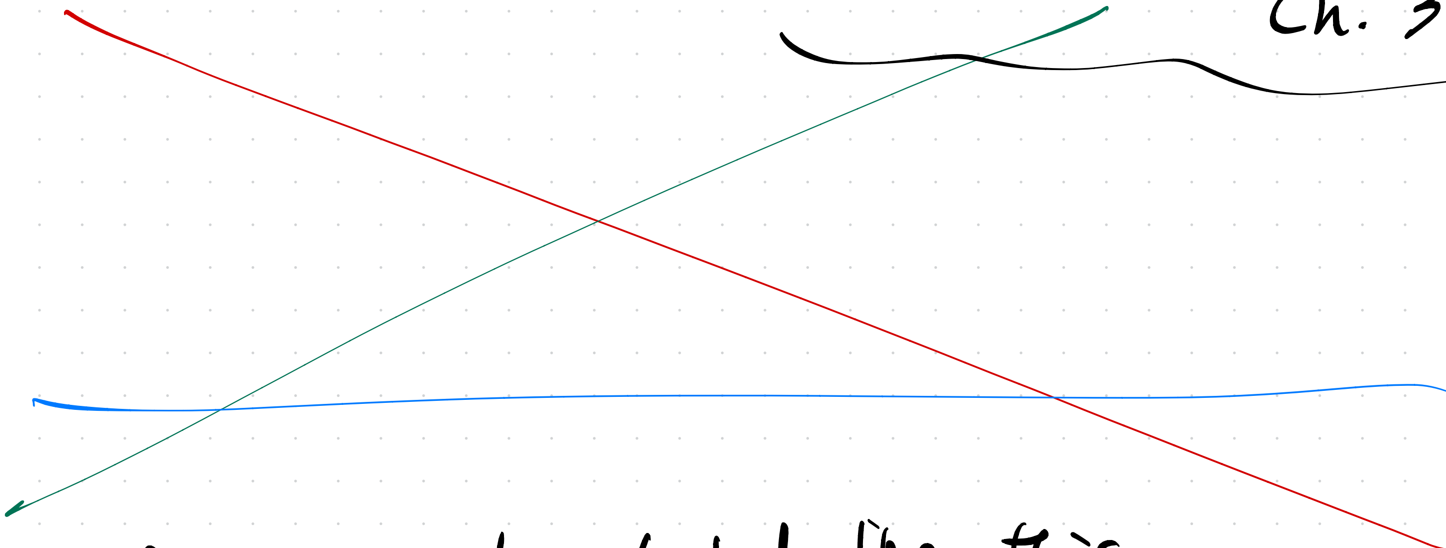


$\vec{x}$
$\vec{b}$
$\vec{y}$
$2\vec{x}$
$\frac{1}{2}\vec{y}$
$2\vec{x}+\frac{1}{2}\vec{y}$

<u>one</u> point of intersection, so <u>one</u> way to write $\vec{b}$ as linear comb. of $\vec{x}$ and $\vec{y}$!



look at the new pictures in Ch. 3.1!



If my system looked like this, then there would be no way to write $\vec{b}$ as a LC of $\vec{x}$ and $\vec{y}$!

Dot product

$\vec{u}, \vec{v} \in \mathbb{R}^n$

same # of components

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

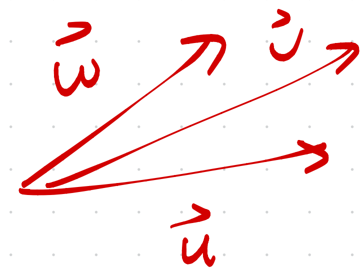
$$\underbrace{\vec{u} \cdot \vec{v}}_{\text{2 vectors}} = \overbrace{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}^{\text{returns a scalar!}} = \vec{v} \cdot \vec{u}$$

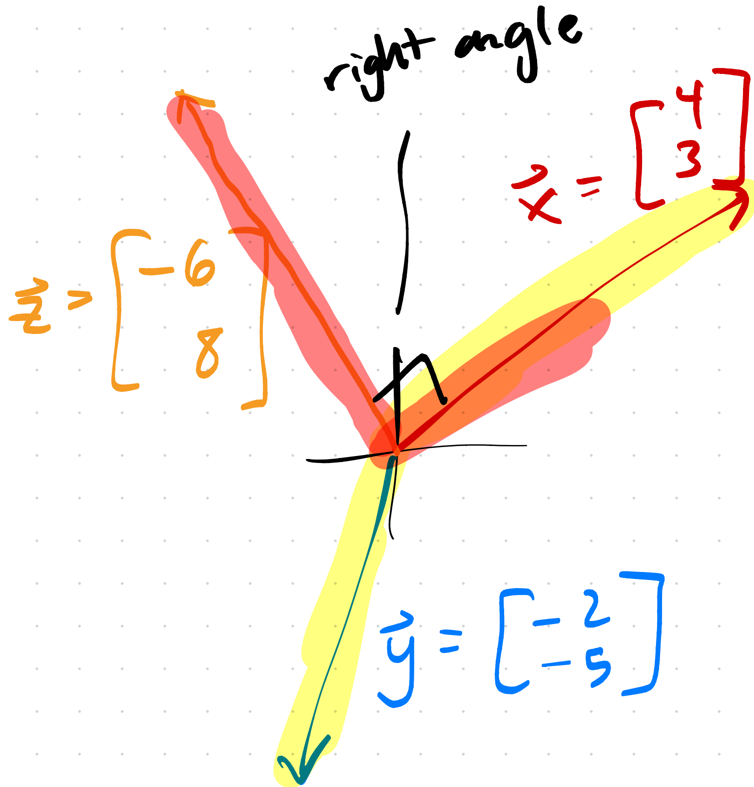
$$\vec{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} -2 \\ -5 \end{bmatrix} \qquad \vec{x} \cdot \vec{y} = (4)(-2) + (3)(-5) = -23$$

$\cancel{\vec{u} \cdot \vec{v} \cdot \vec{w}}$ not defined!

$(\vec{u} \cdot \vec{v})\, \vec{w}$ is defined!

$$(\vec{u} \cdot \vec{v})\, \vec{w} \neq \vec{u}\, (\vec{v} \cdot \vec{w}) = (\vec{v} \cdot \vec{w})\, \vec{u}$$





$$\vec{x} \cdot \vec{y} = -23$$

$$\vec{x} \cdot \vec{x} = (4)(4) + (3)(3) = 25$$

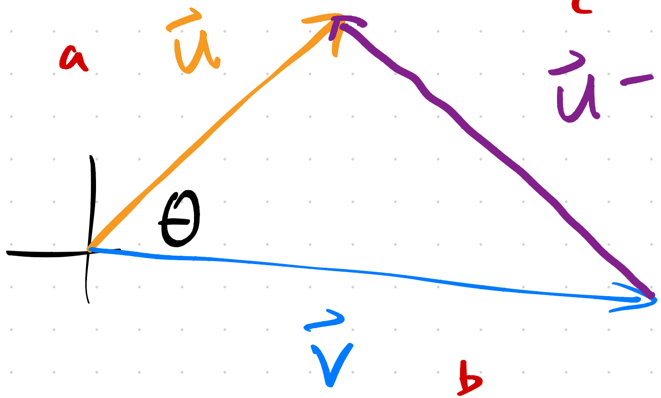
$$\vec{x} \cdot \vec{z} = \vec{z} \cdot \vec{x} = (-6)(4) + (8)(3) = 0$$

"orthogonal"
"perpendicular"

key idea: dot product measures

SIMILARITY

between vectors

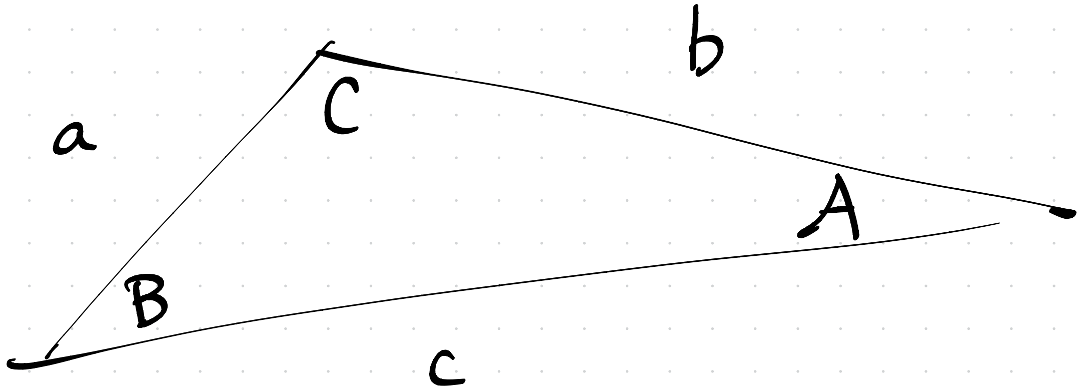


$$\vec{u} - \vec{v} = -\vec{v} + \vec{u}$$

general: $c^2 = a^2 + b^2 - 2ab\cos(C)$

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

Aside: cosine law!



$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

Aside:

$$\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$$

$$\vec{v}\cdot\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= (v_1)(v_1) + (v_2)(v_2) + \cdots + (v_n)(v_n)$$

$$= v_1^2 + v_2^2 + \cdots + v_n^2$$

$$= \|\vec{v}\|^2$$

we've discovered:

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

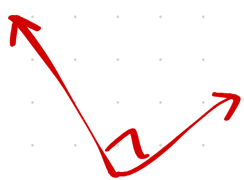
$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v}$$

equal!

$$\cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} \underbrace{-2}_{-2}\|\vec{u}\|\|\vec{v}\|\cos\theta = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} \underbrace{-2}_{-2}\vec{u}\cdot\vec{v}$$

$$\Rightarrow \quad \vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$
$$= u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

"geometric definition"

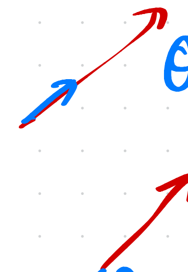


$\cos 90^\circ = 0$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

$$-1 \le \cos\theta \le 1$$



$\theta = 0$, $\cos\theta = 1$

✓ $\theta = 180$, $\cos\theta = -1$

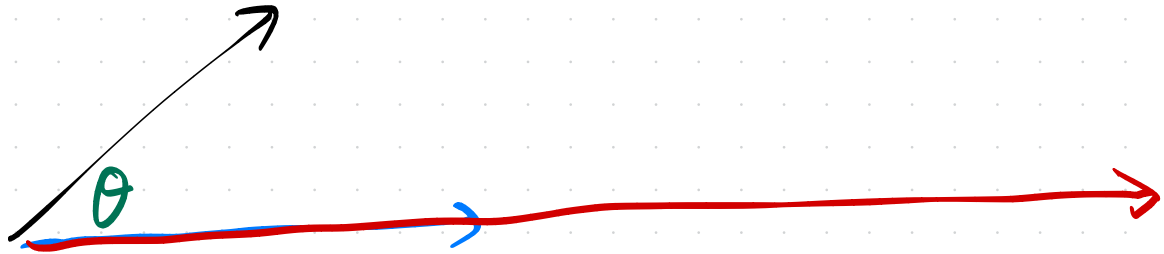
"cosine similarity" of 2 vectors

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \underbrace{\left(\frac{\vec{u}}{\|\vec{u}\|}\right)}_{\text{"unit vector"}} \cdot \left(\frac{\vec{v}}{\|\vec{v}\|}\right)$$

$$\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\frac{\vec{u}}{\|\vec{u}\|} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

Same direction as $\vec{u}$!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Aside:

"default" $\|\vec{v}\|_2 = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$ "L_2"

$\|\vec{v}\|_1 = |v_1| + |v_2| + \dots + |v_n|$ "L_1"

$\|\vec{v}\|_\infty$ = max absolute value of any component "L_∞"

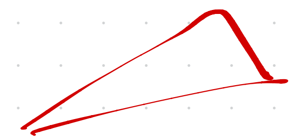
$\vec{v} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

$\|\vec{v}\|_\infty = 3$

Two important inequalities

assume (as usual) the L_2 norm

① Triangle inequality



$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$$

② Cauchy-Schwarz

$$|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\,\|\vec{v}\|$$

ex

Prove that the geometric mean of a, b $\leq$ arithmetic mean of a, b

want to show:

$$\sqrt{ab} \leq \frac{a+b}{2}$$

hint: use C-Z inequality

$$\vec{u} = \begin{bmatrix} \sqrt{a} \\ \ \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} \ \\ \ \end{bmatrix}$$