EEcS 245 Fall 2025

Math for ML

Lecture 6: Dot Product, Projections

→ Read Ch. 2.1-2.2 (2.3 coming soon)

Agenda

① Vectors and linear combinations } Ch. 2.1

② Dot Product
- → Definition
- → Properties
- → Deriving the "other" definition
- → Orthogonality
- → Inequalities

Ch. 2.2

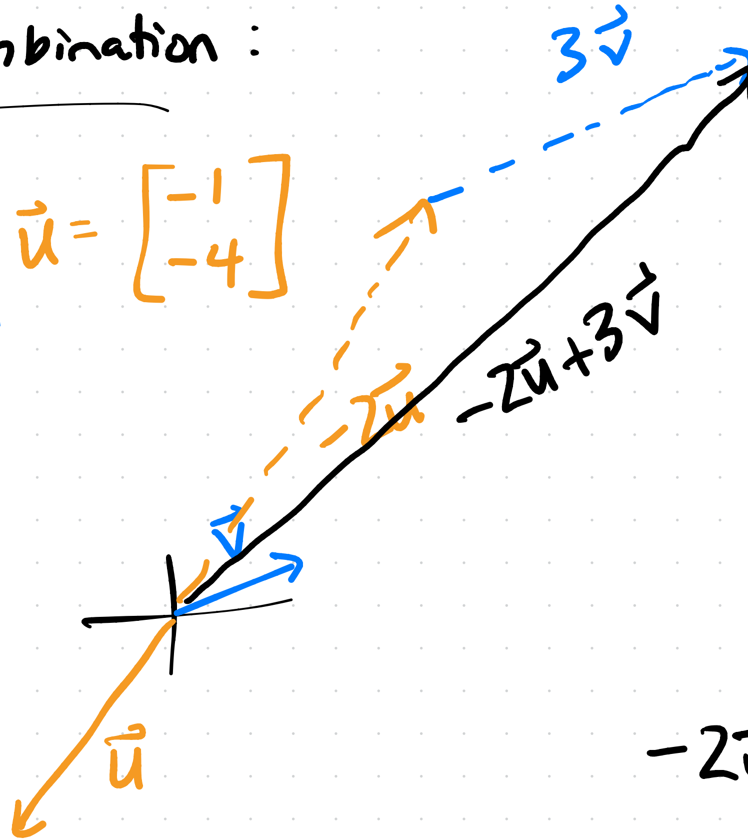
③ The "Approximation Problem" } 2.3 (coming soon!)

Linear combination:

ex/

$\vec{u} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$



$\underbrace{a\vec{u} + b\vec{v}}$

linear combination of $\vec{u}$, $\vec{v}$

$-2\vec{u} + 3\vec{v}$

$\mathbb{R}^2$: set of all vectors with 2 real numbers (i.e. 2 components)

$$\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

Write $\begin{bmatrix} 9 \\ -16 \\ -1 \end{bmatrix}$ as a linear combination of $\vec{x}$ and $\vec{y}$

$$c\vec{x} + d\vec{y} = \begin{bmatrix} 9 \\ -16 \\ -1 \end{bmatrix}$$

$$c\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + d\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ -16 \\ -1 \end{bmatrix}$$

$$\begin{aligned} 3c + d &= 9 \\ -c + 4d &= -16 \\ 2c + 3d &= -1 \end{aligned}$$

$$3c + d = 9 \quad (1)$$
$$-c + 4d = -16 \quad (2)$$
$$2c + 3d = -1 \quad (3)$$

satisfies (3)

even though we didn't use it!

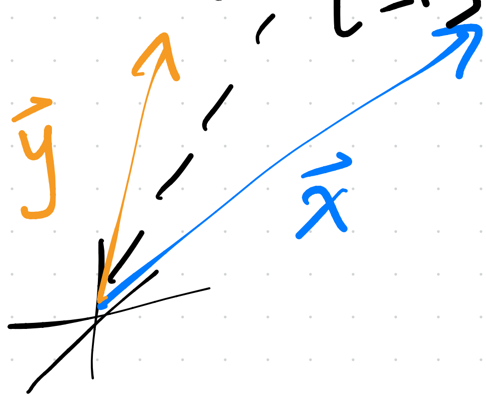
$$3c + d = 9 \quad (1)$$
$$-3c + 12d = -48 \quad (2)\times 3$$

$$13d = -39 \Rightarrow d = -3$$

$$3c - 3 = 9 \rightarrow c = 4$$

$\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

Write $\begin{bmatrix} 9 \\ -16 \\ -1 \end{bmatrix}$ as a linear combination of $\vec{x}$ and $\vec{y}$



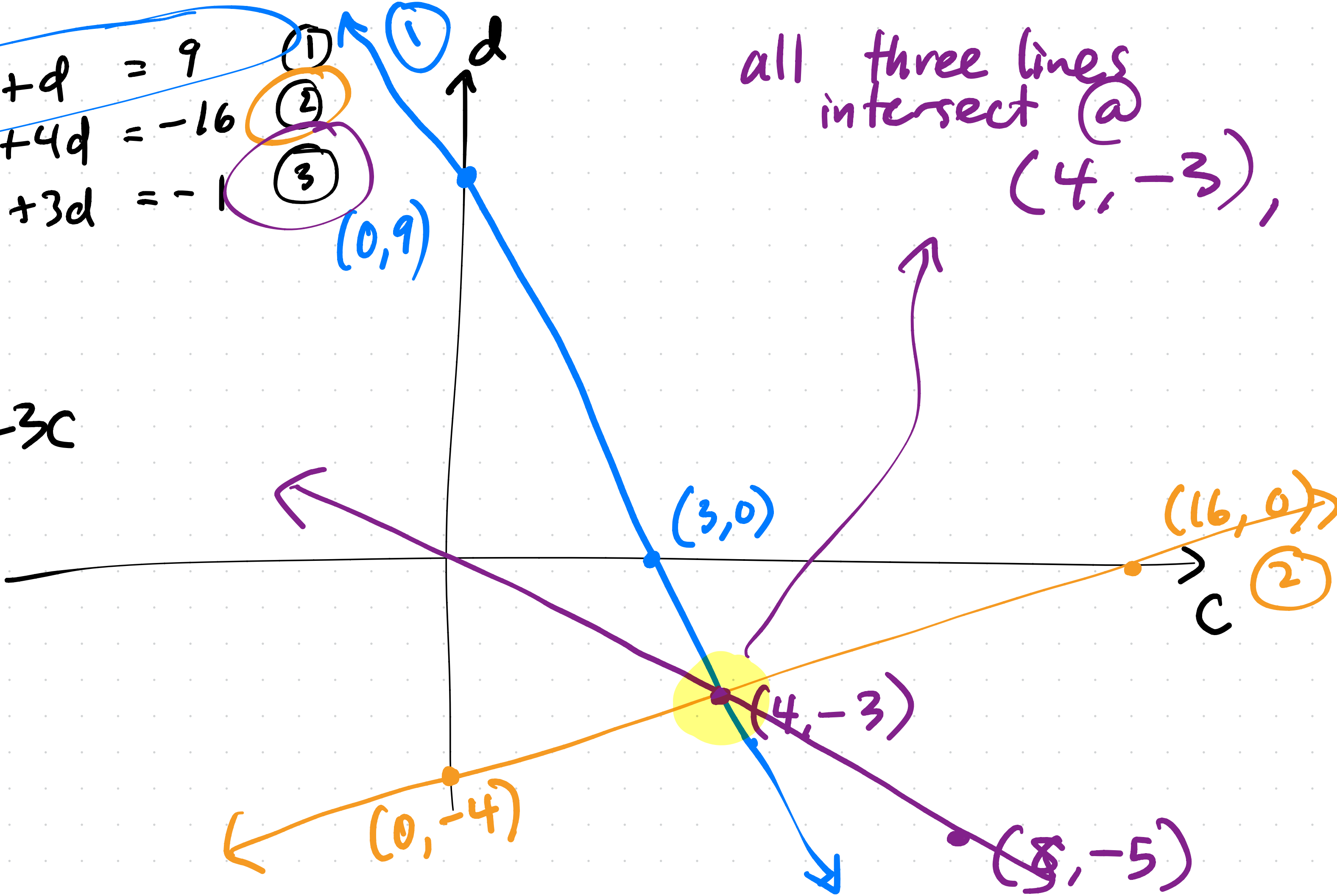
Since $c = 4$, $d = -3$,

$$\begin{bmatrix} 9 \\ -16 \\ -1 \end{bmatrix} = 4\vec{x} - 3\vec{y}$$

The set of all linear combinations of $\vec{x}$ and $\vec{y}$ is a <u>plane</u> in $\mathbb{R}^3$

$$3c + d = 9 \quad (1)$$
$$-c + 4d = -16 \quad (2)$$
$$2c + 3d = -1 \quad (3)$$

$$d = 9 - 3c$$



Dot Product

$$\underbrace{\vec{u}, \vec{v} \in \mathbb{R}^n}_{\text{"same \# of components"}}$$

$$\overbrace{\vec{u} \cdot \vec{v}}^{\text{two vectors}} \qquad \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \underbrace{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}_{\text{scalar!}}$$

$$\vec{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

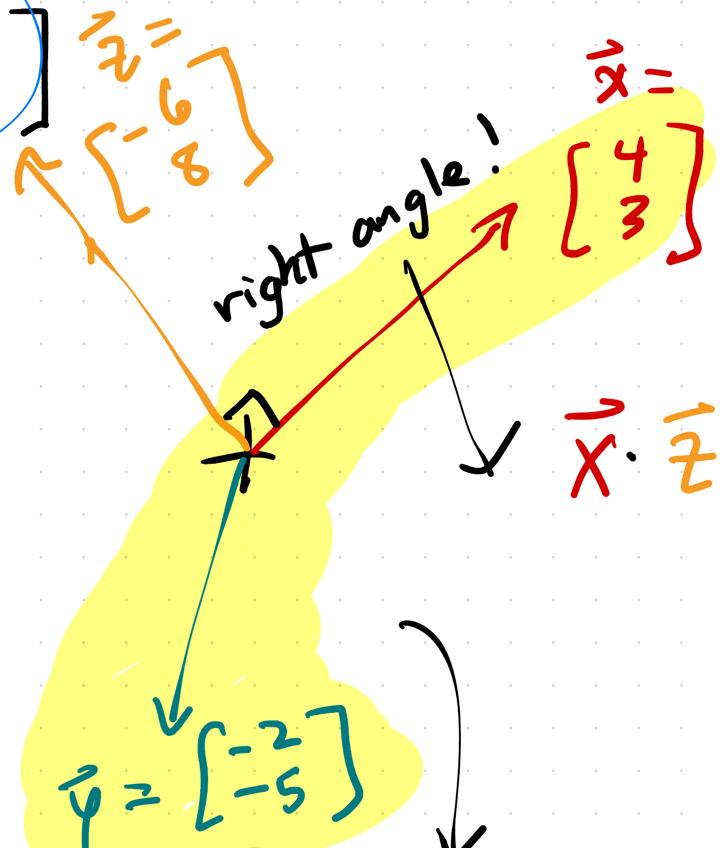
$$\vec{z} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$$

$$\rightarrow \vec{x} \cdot \vec{y} = (4)(-2) + (3)(-5)$$
$$= -8 - 15 = -23$$

$$\rightarrow \vec{x} \cdot \vec{z} = 4(-6) + 3(8) = 0$$

$\vec{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$

$\vec{z} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$

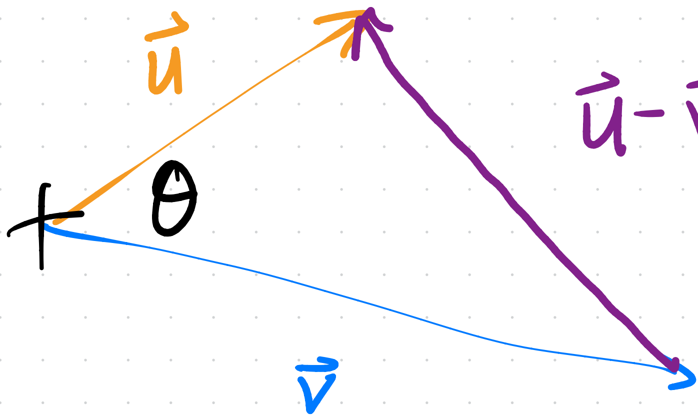


$\vec{x} \cdot \vec{z} = 0$

$\vec{x} \cdot \vec{x}$
$= 4(4) + 3(3)$
$= 25$

$\vec{x} \cdot \vec{y} = -23$ → point in opposite (ish) directions

key idea: dot product measures similarity



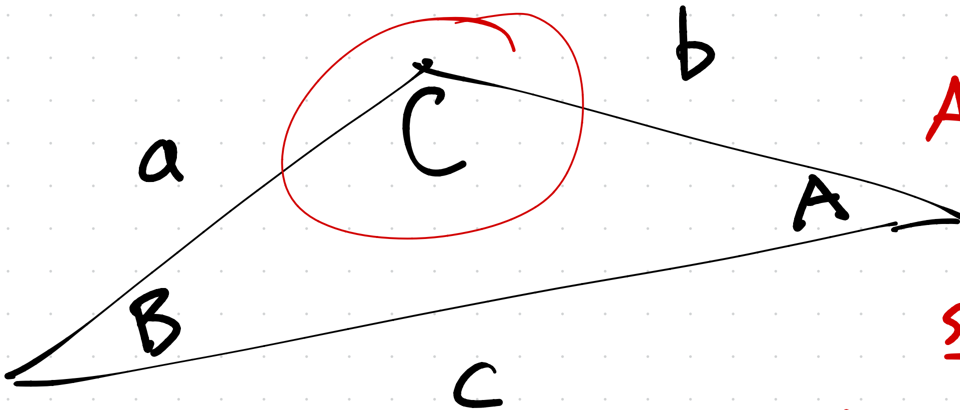
cosine law

$$c^2 = a^2 + b^2 - 2ab\cos C$$

applying cosine law:

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

Aside



a, b, c side lengths

A, B, C angles

sin law

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

cosine law

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

applying cosine law:

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

In general, $\vec{v}\cdot\vec{v} = \sum_{i=1}^{n} v_i^2 = \|\vec{v}\|^2$

Expand

commutative

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v}$$

first

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

second

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\vec{u}\cdot\vec{v}$$

Both are equal!

$$\cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} \underset{-2}{-2}\|\vec{u}\|\,\|\vec{v}\|\cos\theta = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} \underset{-2}{-2}\,\vec{u}\cdot\vec{v}$$

$$\|\vec{u}\|\,\|\vec{v}\|\cos\theta = \vec{u}\cdot\vec{v}$$

$$\cos(90^\circ) = 0$$

since $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

we know $\vec{u}, \vec{v}$ orthogonal (perpendicular)

<u>when</u> $\vec{u} \cdot \vec{v} = 0$.

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

dot product of $\left(\frac{\vec{u}}{\|\vec{u}\|}\right) \cdot \left(\frac{\vec{v}}{\|\vec{v}\|}\right)$

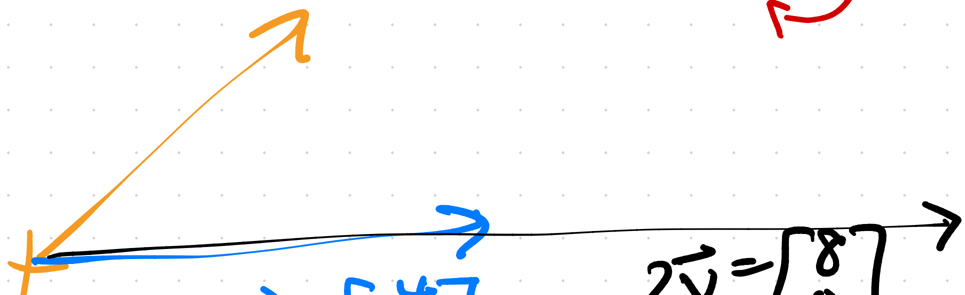
$$-1 \le \cos\theta \le 1$$

"cosine similarity" of $\vec{u}, \vec{v}$

$$\vec{u} \cdot \vec{v} = 12$$

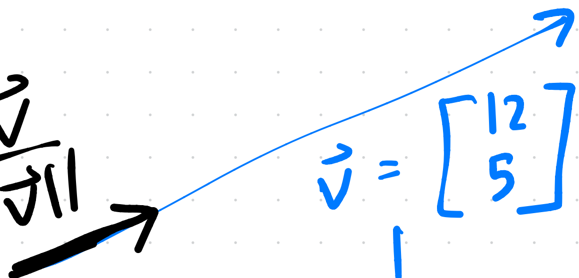
$$\vec{u} \cdot (2\vec{v}) = 2\vec{u} \cdot \vec{v} = 24$$

$$\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



$$\vec{v} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \qquad 2\vec{v} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

"Unit" vector : $\|\vec{v}\| = 1$



$$\frac{\vec{v}}{\|\vec{v}\|}$$

$$\vec{v} = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$$

$$\downarrow$$

$$\frac{\vec{v}}{\|\vec{v}\|} = \begin{bmatrix} 12/13 \\ 5/13 \end{bmatrix}$$

5, 12, 13

7, 24, 25

11, 60, 61

$\vdots$

unit vectors to describe <u>direction</u>.

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

"default" norm

L_2 norm

<u>alternative norms</u>

"L_1 norm"

$$\|\vec{v}\|_1 = |v_1| + |v_2| + \dots + |v_n|$$