EECS 245, Winter 2026

LEC 7 Projections

→ Read: Ch. 3.4, maybe 4.1 (if time)

Agenda

- Recap: important inequalities
- The "approximation problem"
  - → Motivation
  - → Solution (orthogonal projections)
  - → Examples
- If time: span (Ch. 4.1)

Announcements

- HW 3 due tonight (I have OH in 4174 Leinweber after)
- HW 4 released tonight
- Check out the Grade Report on Gradescope!

① Triangle inequality

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

all vector norms, not just L_2!

② Cauchy-Schwarz inequality

$$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|$$

only applies to L_2!

Activity: Prove that the arithmetic mean of a, b $\geq$ geometric mean of a, b using Cauchy-Schwarz

$$\sqrt{ab} \leq \frac{a+b}{2}$$

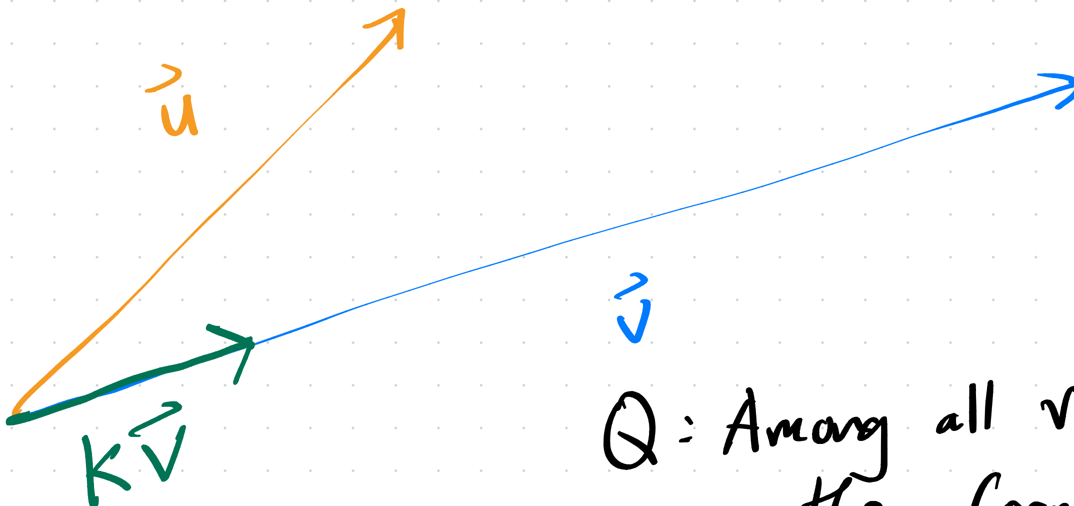
$$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\| \qquad \vec{u} = \begin{bmatrix} \sqrt{a} \\ \sqrt{b} \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} \sqrt{b} \\ \sqrt{a} \end{bmatrix}$$

$$\left|\sqrt{ab} + \sqrt{ba}\right| \leq \underbrace{\sqrt{(\sqrt{a})^2 + (\sqrt{b})^2}}_{\|\vec{u}\|}\sqrt{(\sqrt{a})^2 + (\sqrt{b})^2}$$

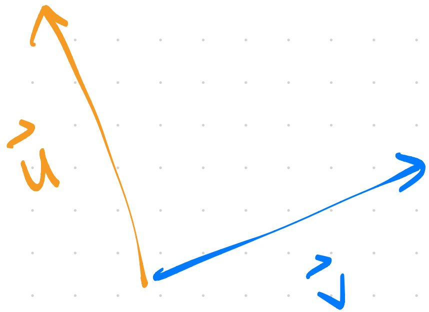
$$2\sqrt{ab} \leq a+b \longrightarrow \sqrt{ab} \leq \frac{a+b}{2} \quad \checkmark$$

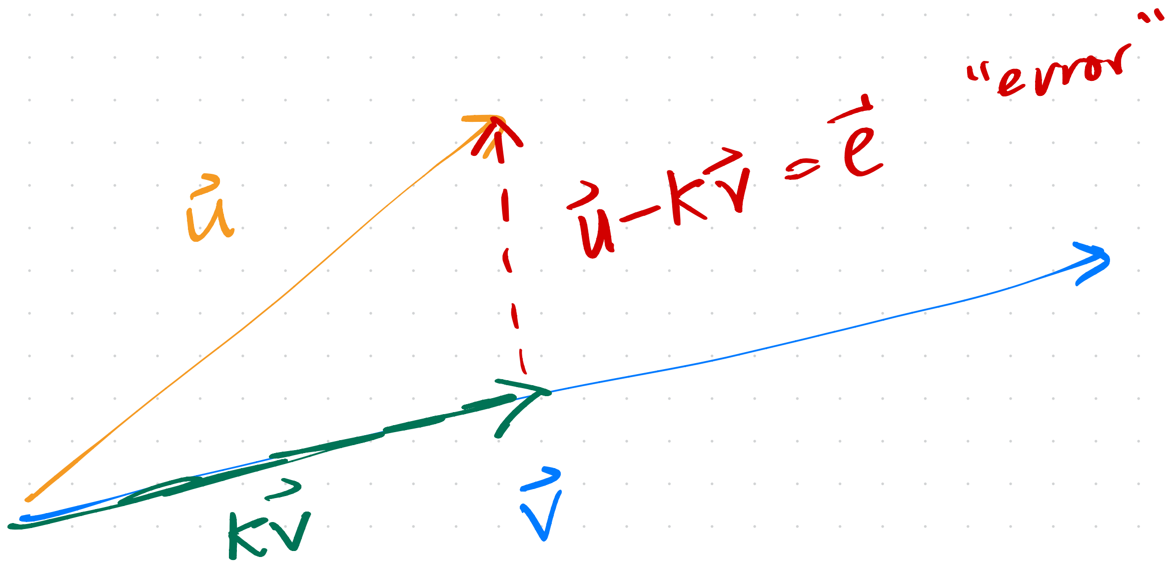
Chapter 3.4

$$\| \vec{u} - k\vec{v} \|^2$$



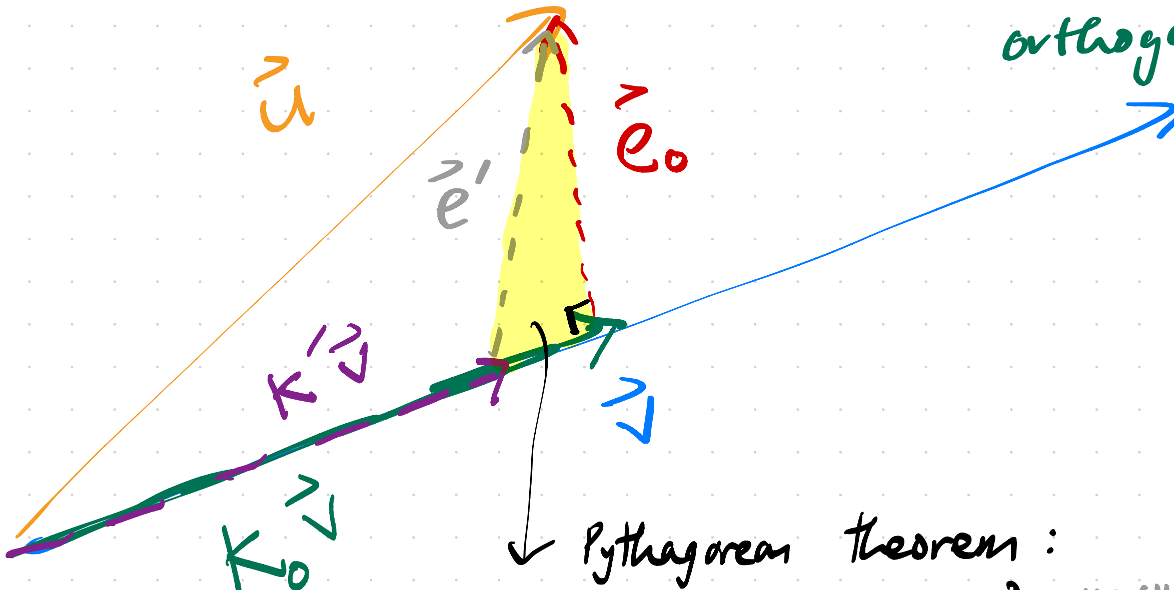
Q: Among all vectors of the form $k\vec{v}$, which is closest to $\vec{u}$?


$\vec{u}$
$\vec{v}$



$\vec{u}$
$\vec{u} - k\vec{v} = \vec{e}$
"error"
$k\vec{v}$
$\vec{v}$

suppose K_0 makes $\vec{e}_0 = \vec{u} - K_0\vec{v}$ orthogonal to $\vec{v}$

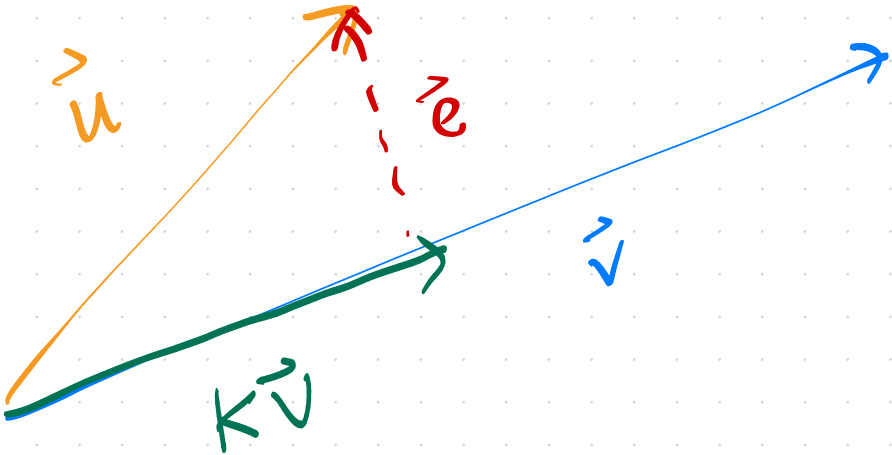


suppose k' is some other scalar other than K_0

Pythagorean theorem:

$$\|\vec{e}_0\|^2 + \|K_0\vec{v} - k'\vec{v}\|^2 = \|\vec{e}\,'\|^2$$

$$\|\vec{e}_0\|^2 + \underbrace{\|(K_0 - k')\vec{v}\|^2}_{>0} = \|\vec{e}\,'\|^2$$



The way to make $k\vec{v}$ as close to $\vec{u}$ as possible is to make

$\vec{u} - k\vec{v} = \vec{e}$

orthogonal to $\vec{v}$!

$$\vec{e} \cdot \vec{v} = 0$$

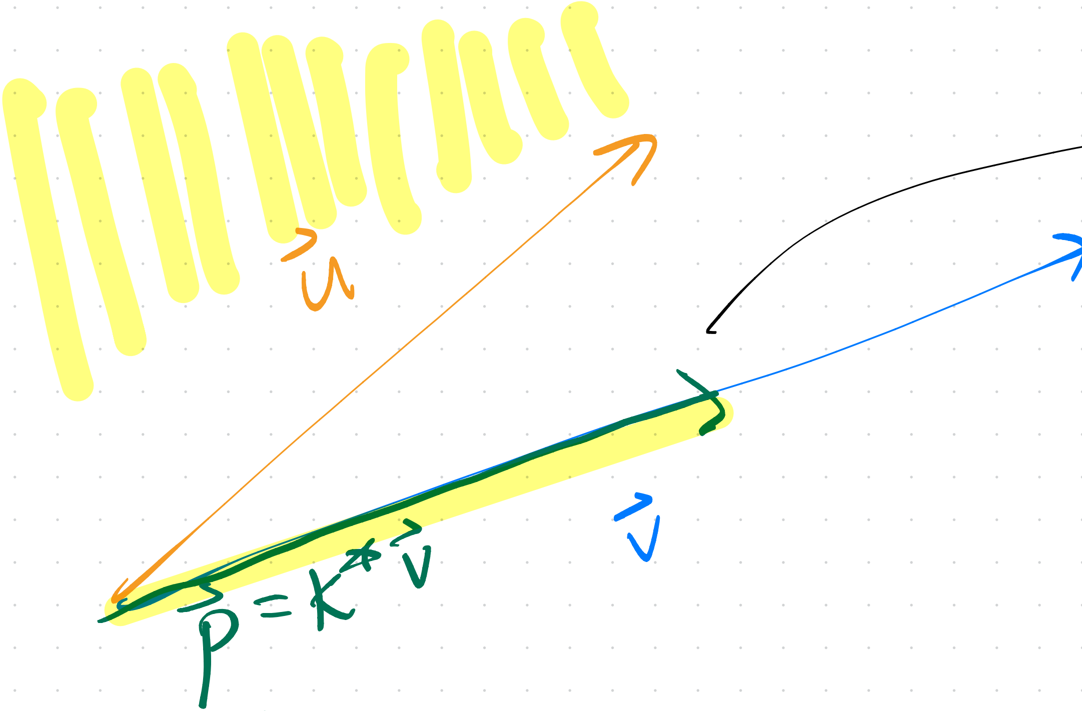
$$(\vec{u} - k\vec{v}) \cdot \vec{v} = 0$$

$$\vec{u} \cdot \vec{v} - (k\vec{v}) \cdot \vec{v} = 0$$

$$\vec{u} \cdot \vec{v} - k(\vec{v} \cdot \vec{v}) = 0$$

$$k^* = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

optimal k!



"shadow of u" onto $\vec{v}$

"orthogonal projection" of $\vec{u}$ onto $\vec{v}$

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

① Find $\vec{p}$, projection of $\vec{u}$ onto $\vec{v}$

② Find error vector, verify that it is orthogonal to $\vec{v}$

③ Find the length of error vector

① $$k^* = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}} = \frac{5}{3}$$

$$\vec{p} = \frac{5}{3}\vec{v} = \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix}$$

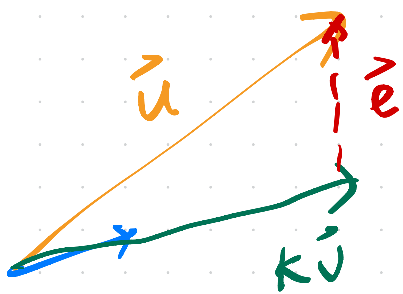
② $$\vec{e} = \vec{u} - k^*\vec{v} = \vec{u} - \vec{p} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix}$$

$$= \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

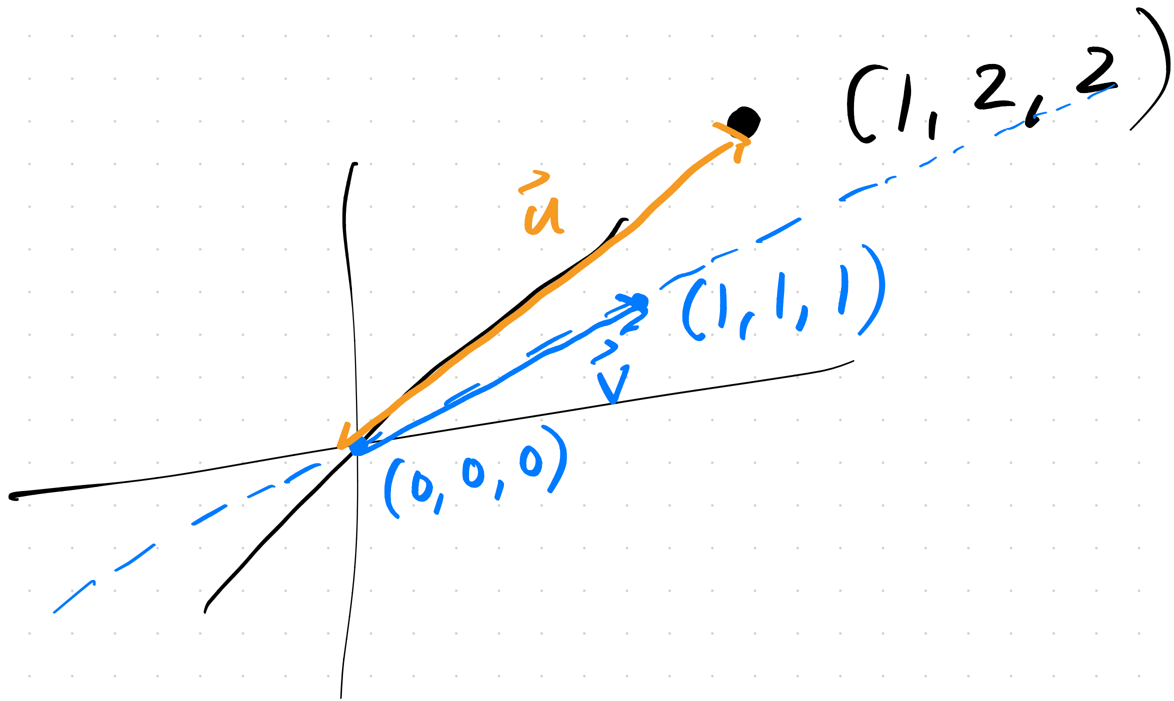
$$\vec{e} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{e} \cdot \vec{v} = 0 \checkmark$$

3 $$\|\vec{e}\| = \sqrt{\left(-2/3\right)^2 + \left(1/3\right)^2 + \left(1/3\right)^2} = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}}$$

$$= \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}$$



"projection error"



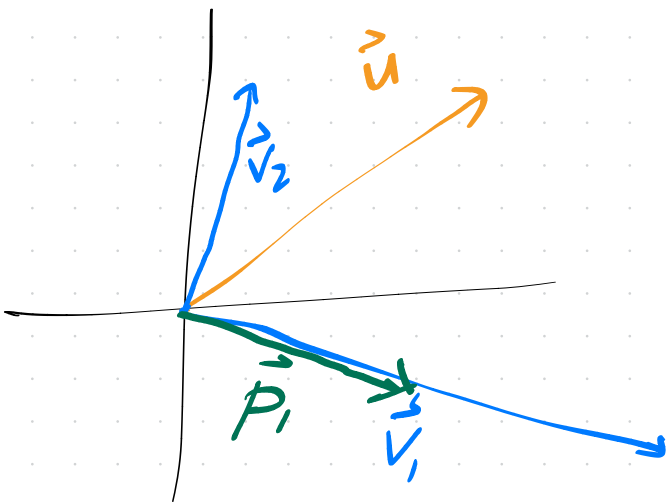
(1, 2, 2)
$\vec{u}$
$\vec{v}$
(1, 1, 1)
(0, 0, 0)

$\vec{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ $\qquad$ $\vec{v}_1 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ $\qquad$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Write $\vec{u}$ as a <u>linear combination</u> of $\vec{v}_1, \vec{v}_2$

$$\vec{u} = \underbrace{\frac{1}{5}}_{a}\vec{v}_1 + \underbrace{\frac{9}{5}}_{b}\vec{v}_2$$

$$\begin{cases} 6a + b = 3 \\ -2a + 3b = 5 \end{cases}$$



$$\vec{p}_1 = \left(\frac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 = \left(\frac{8}{40}\right)\vec{v}_1 = \left(\frac{1}{5}\right)\vec{v}_1$$

$\vec{v}_1 \cdot \vec{v}_2 = 0$

$$\vec{u} = a\vec{v}_1 + b\vec{v}_2$$

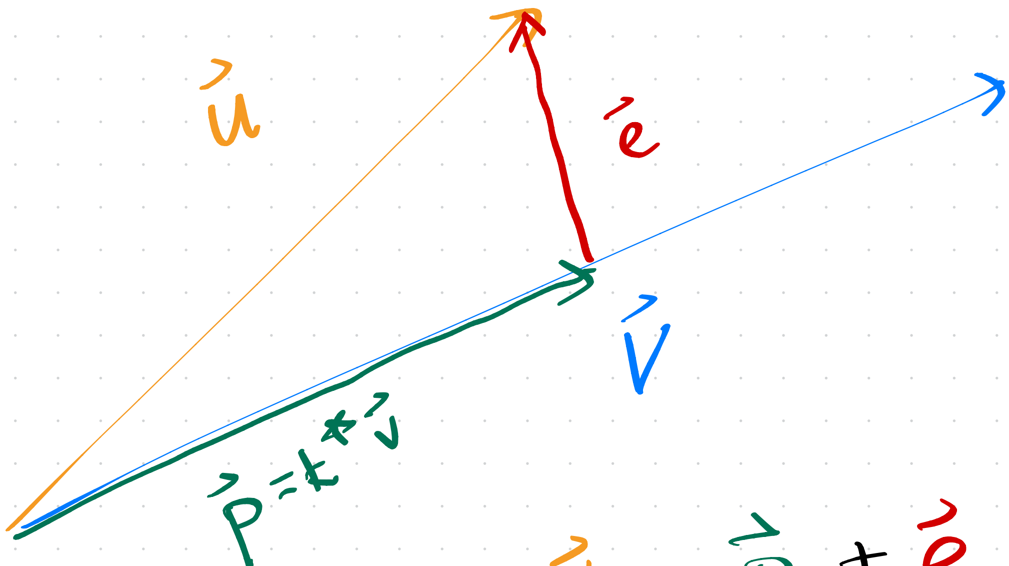
idea: dot product of both sides with $\vec{v}_1$!

$$\vec{u} \cdot \vec{v}_1 = a\,\vec{v}_1 \cdot \vec{v}_1 + \underbrace{b\,\vec{v}_2 \cdot \vec{v}_1}_{0}$$

$$\vec{u} \cdot \vec{v}_1 = a(\vec{v}_1 \cdot \vec{v}_1)$$

$$\Rightarrow \quad a = \frac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}$$

this is just k^*!
(when projecting $\vec{u}$ onto $\vec{v}_1$.)



"orthogonal decomposition" of $\vec{u}$

$$\vec{u} = \underbrace{\vec{p}} + \underbrace{\vec{e}}$$

components are orthogonal!