EECS 245 Fall 2025

Math for ML

Lecture 7: Projections and Span

→ Read: 2.3 (new), 2.4 (coming soon)

Agenda

1. Important (in)equalities from last class
2. The "Approximation Problem"
   - → Motivation
   - → Solution (Orthogonal Projections)
   - → Many examples

   Ch. 2.3 (new!)
3. Span } start Ch. 2.4
   - → Equation of a line in $\mathbb{R}^n$

Some updates

→ Thank you for giving feedback on Homework 3!

→ Some changes:

→ Updated lecture recordings (moving forward) to not cut off the last few minutes

→ Future homeworks will be due on Thursdays (more time for OH)

Inequalities (Ch. 2.2)

Triangle inequality $\|\vec{u}\|$ $\|\vec{v}\|$

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$$

Cauchy-Schwarz

$$|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\,\|\vec{v}\|$$

Cauchy-Schwarz

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|$$

scalars

Prove that the geometric mean of a, b is $\leq$ arithmetic mean of a, b

$$\Downarrow$$

$$\sqrt{ab} \leq \frac{a+b}{2}$$

Hint:

$$\vec{u} = \begin{bmatrix} \sqrt{a} \\ \sqrt{b} \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

???

Prove $\sqrt{ab} \le \dfrac{a+b}{2}$

AM-GM-HM inequality

$AM \ge GM \ge HM$

harmonic

$\vec{u} = \begin{bmatrix} \sqrt{a} \\ \sqrt{b} \end{bmatrix}$ $\vec{v} = \begin{bmatrix} \sqrt{b} \\ \sqrt{a} \end{bmatrix}$

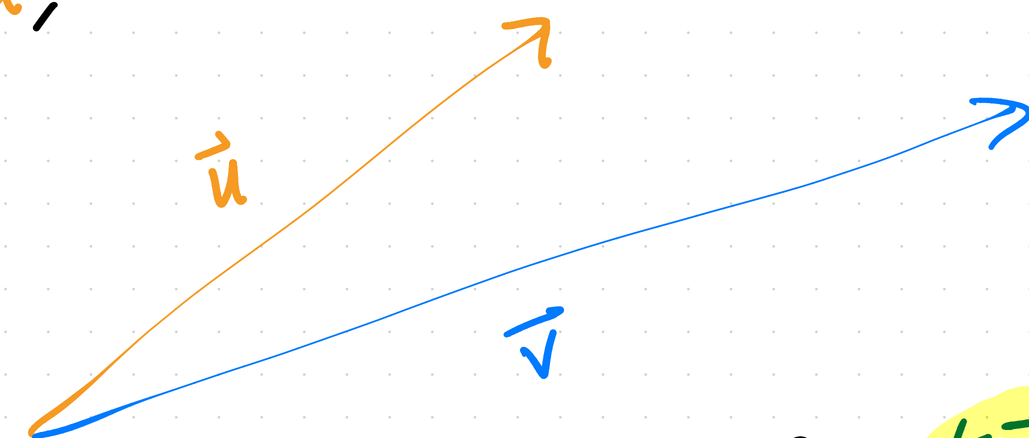
C-Z: $|\vec{u} \cdot \vec{v}| \le \|\vec{u}\| \, \|\vec{v}\|$

$$2\sqrt{ab} \le \underbrace{\sqrt{(\sqrt{a})^2 + (\sqrt{b})^2}}_{\sqrt{a+b}} \sqrt{(\sqrt{b})^2 + (\sqrt{a})^2}$$

$(\sqrt{a+b})^2 = a+b$

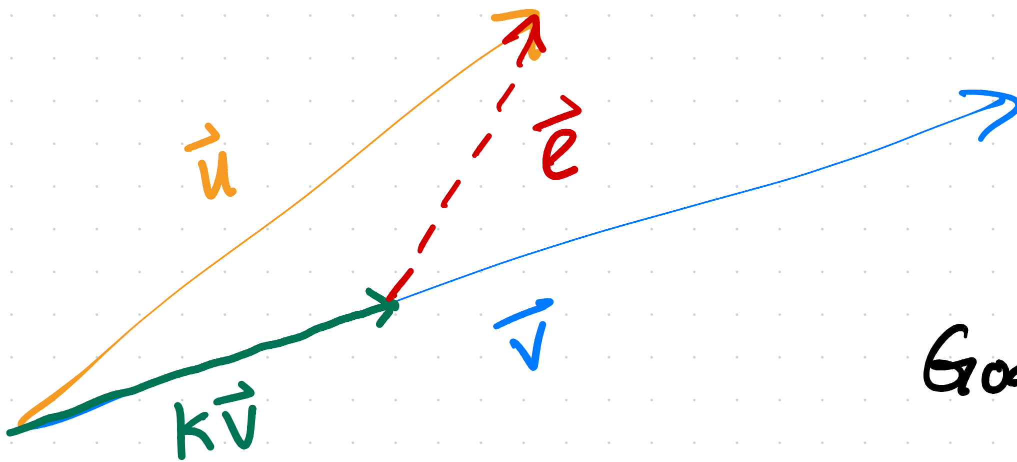
The Approximation Problem (2.3)

$\vec{u}, \vec{v} \in \mathbb{R}^n$



a scalar multiple of $\vec{v}$

Goal: Of all vectors of the form $k\vec{v}$, which is "closest" to $\vec{u}$?



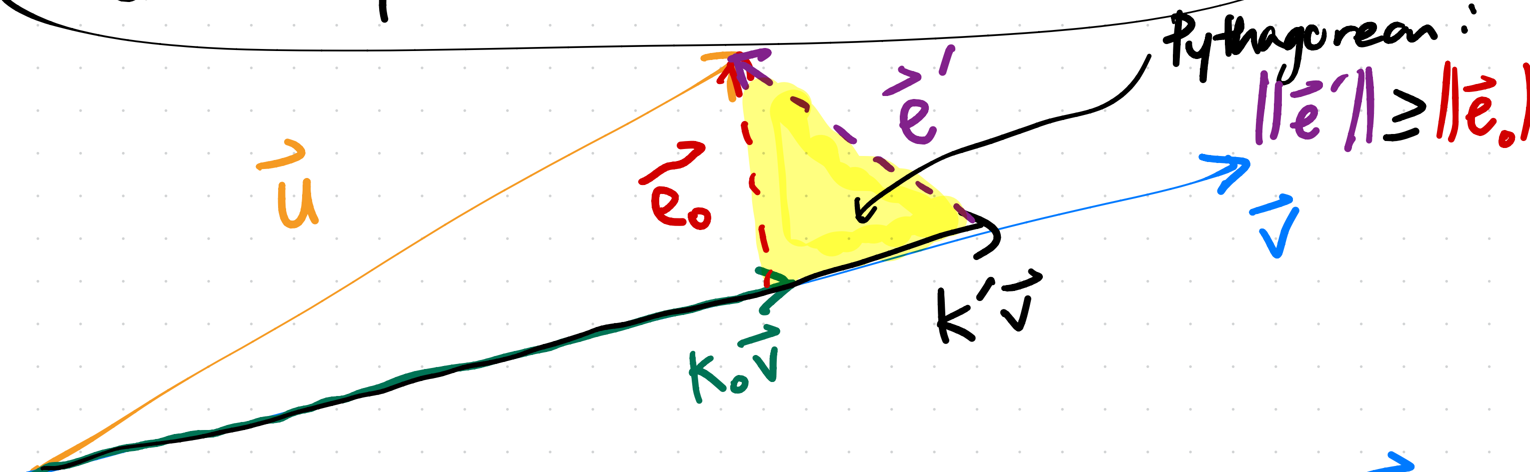
$$\underbrace{\vec{e}}_{\text{error vector}} = \vec{u} - k\vec{v}$$

Goal: Minimize

$$\|\vec{e}\| = \underbrace{\|\vec{u} - k\vec{v}\|}_{\text{function of } k\text{!}}$$

Guess: The best k is the one where $\vec{e}$ is orthogonal to $\vec{v}$

Can we prove that we're right?



Let k_0 be the k such that $\vec{e}_0$ orthogonal to $\vec{v}$

Let k' be some other $k \neq k_0$, error $\vec{e}'$

Shown that the best k, k^*, makes

$\vec{e}$ orthogonal to $\vec{v}$

How do we find k^*?

$$\vec{e} \cdot \vec{v} = 0$$

$$(\vec{u} - k^*\vec{v}) \cdot \vec{v} = 0$$

$$\vec{u} \cdot \vec{v} - (k^*\vec{v}) \cdot \vec{v} = 0$$

$$\vec{u} \cdot \vec{v} - k^*(\vec{v} \cdot \vec{v}) = 0$$

$$\Rightarrow k^* = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

$\|\vec{v}\|^2$

The orthogonal projection of $\vec{u}$ onto $\vec{v}$

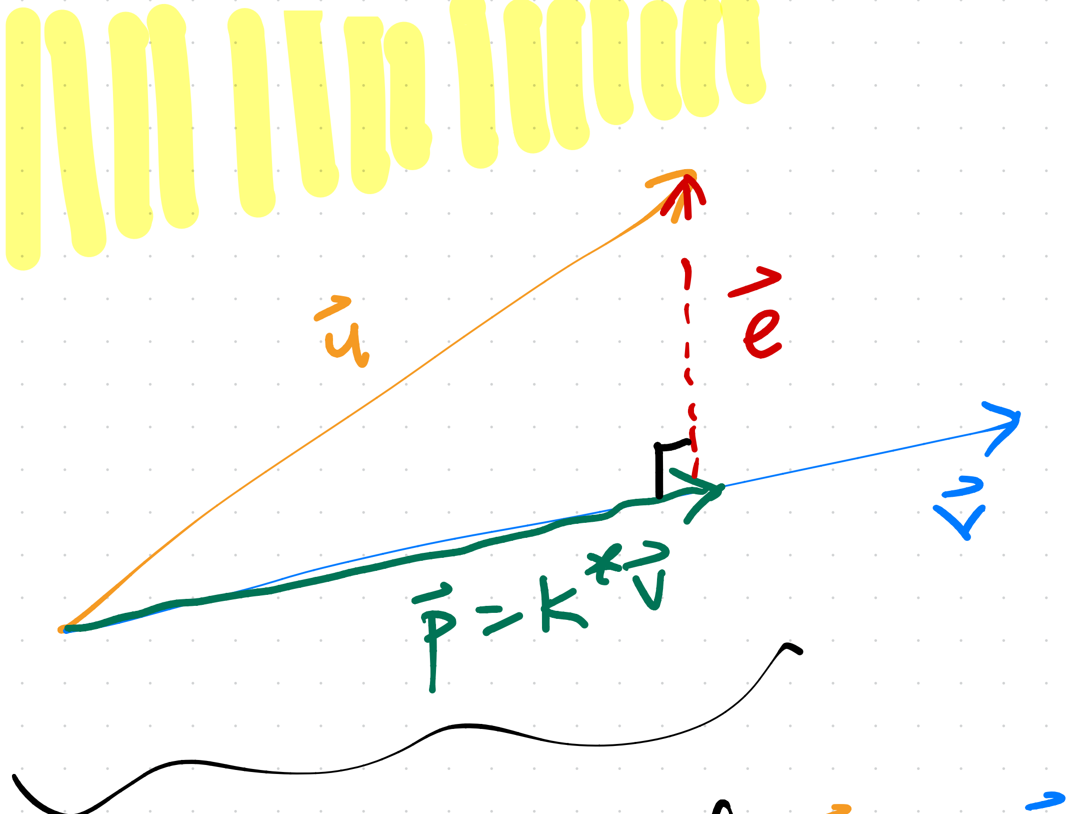
is the vector

$$\vec{p} = \underbrace{\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)}_{k^*}\vec{v}$$

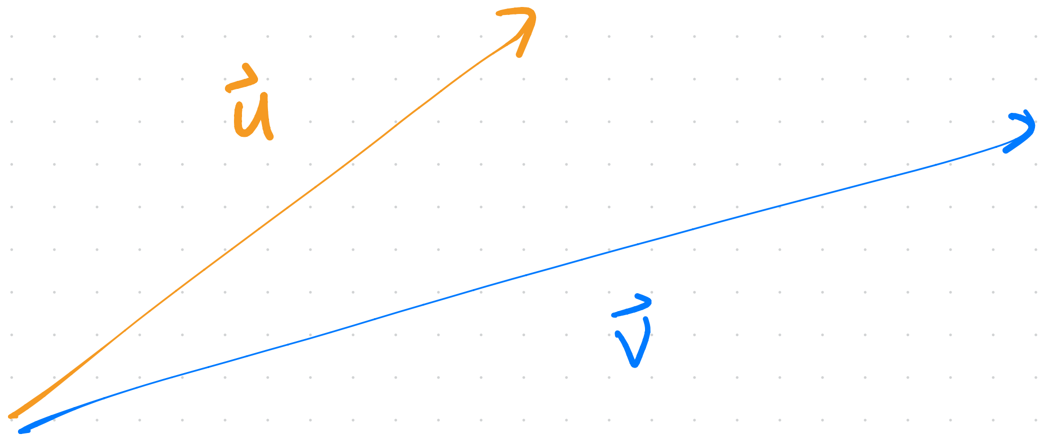
of all vectors of the form $k\vec{v}$,

$\vec{p}$ has the shortest error vector,

$\vec{e} = \vec{u} - \vec{p}$.



$\vec{p}$ is shadow of $\vec{u}$ on $\vec{v}$


$\vec{u}$
$\vec{v}$

$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

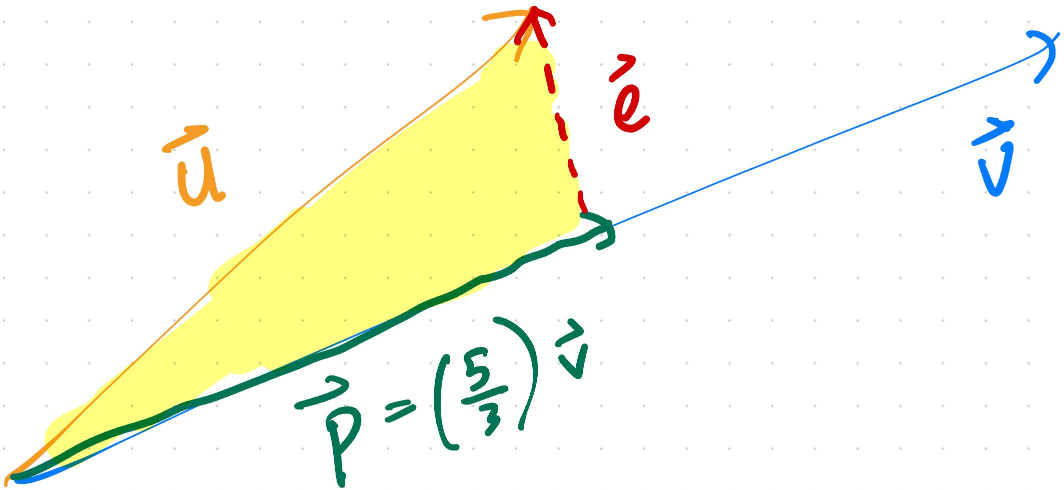
1. find the projection of $\vec{u}$ onto $\vec{v}$
2. find the error vector
3. show that the error vector is orthogonal to $\vec{v}$

① $\vec{p} = \left(\dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v}$

$= \dfrac{5}{3} \vec{v} = \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix}$

② $\vec{e} = \vec{u} - \vec{p} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

③ $\vec{e} \cdot \vec{v} = 0 \rightarrow \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$ ✓



$$\vec{u} = \underbrace{\vec{p}}_{\text{parallel to } \vec{v}} + \underbrace{\vec{e}}_{\text{orthogonal to } \vec{v}}$$

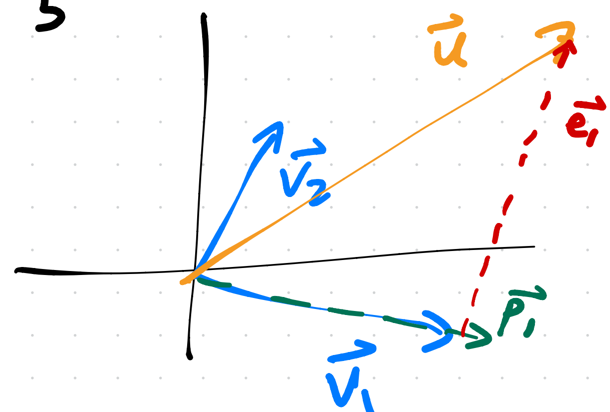
"orthogonal decomposition" of $\vec{u}$ with respect to $\vec{v}$

Write $\vec{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ as a linear combination of

$\vec{v}_1 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$a_1 \begin{bmatrix} 6 \\ -2 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{aligned} 6a_1 + a_2 &= 3 \\ -2a_1 + 3a_2 &= 5 \end{aligned} \longrightarrow a_1 = \frac{1}{5} \quad a_2 = \frac{9}{5}$$



$$\vec{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\frac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \frac{8}{40} = \frac{1}{5} \quad a_1$$

$$\frac{\vec{u} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} = \frac{18}{10} = \frac{9}{5} \quad a_2$$

Span

n: # of dimensions

d: # of vectors

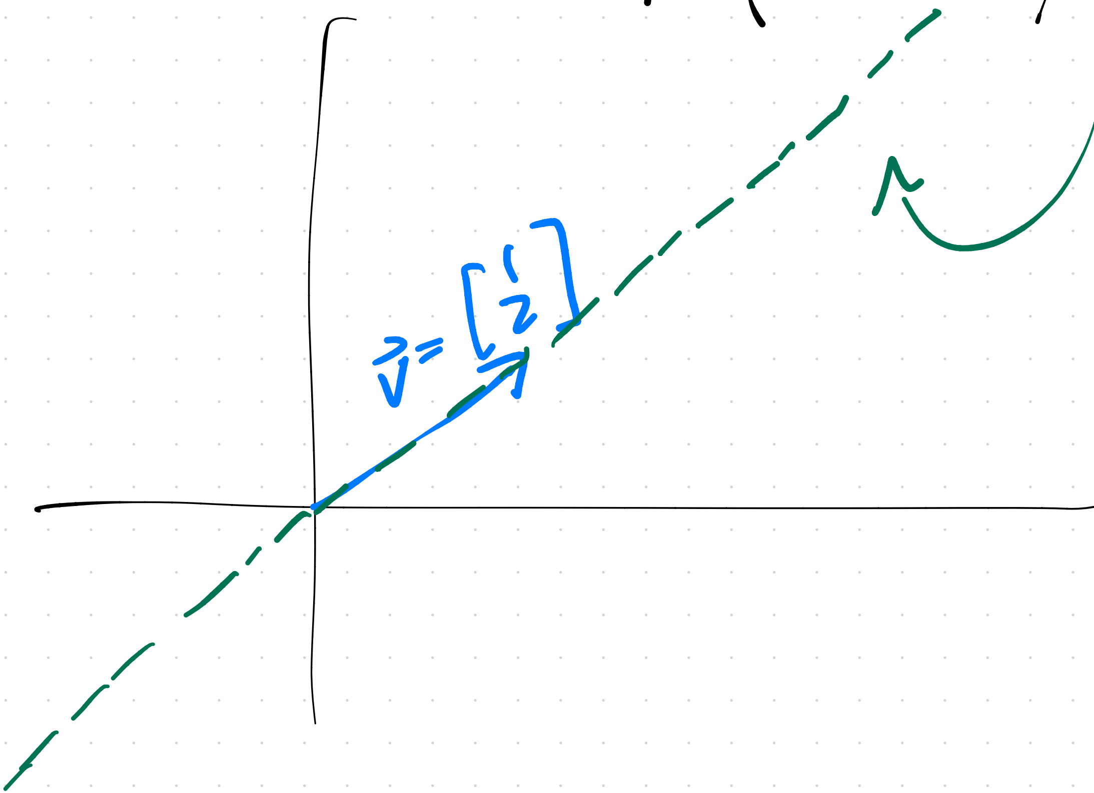
The span of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d \in \mathbb{R}^n$ is the set of all possible linear combinations of those vectors.

input: "spanning set"

$$\operatorname{span}\left(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\}\right) = \left\{ a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_d\vec{v}_d \;\middle|\; a_1, a_2, \ldots, a_d \in \mathbb{R} \right\}$$

Span of a single vector

$\text{span}\left(\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}\right)$ is a line



span includes

(1, 2)

AND

(0, 0)

Parametric equation of line

$$L = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 2 \\ -5 \\ 2 \end{bmatrix} \quad , \quad t \in \mathbb{R}$$