EECS 245, Winter 2026

LEC 8 Span and Linear Independence

→ Read Ch. 4.1 & 4.2

→ 4.4 also helpful for added geometric details

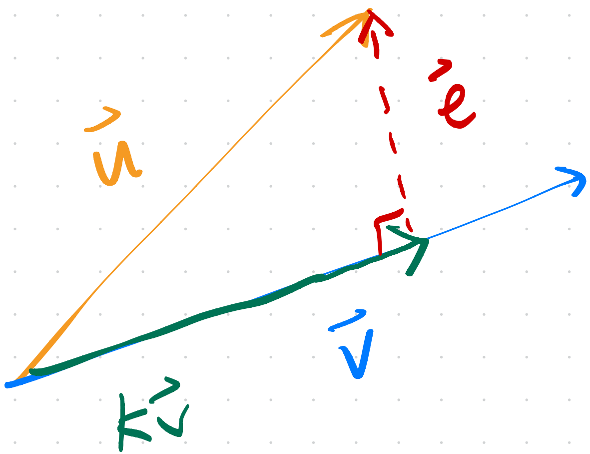
Agenda

Ch. 4 is the last chapter in scope for MT 1!

- Quick recap of orthogonal projections
- The **span** of a set of vectors
  - 4.1
    - 1 vector in $\mathbb{R}^2$? $\mathbb{R}^3$? $\mathbb{R}^n$?
    - 2 vectors?
    - In general, given d vectors in $\mathbb{R}^n$, what can we make with their linear combinations?
- Linear independence } 4.2

Announcements

- Thank you for the feedback on HW 3!
- HW 4 due Monday
- Lab 5 starts tomorrow
- Check out Grade Report on Gradescope
- Practice MT1 next Friday!



$$\|\vec{u} - k\vec{v}\| = \|\vec{e}\|$$

"orthogonal projection"

the **span** of $\underbrace{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d}_{\text{"spanning set"}}$ is the set of all possible linear combinations of those vectors

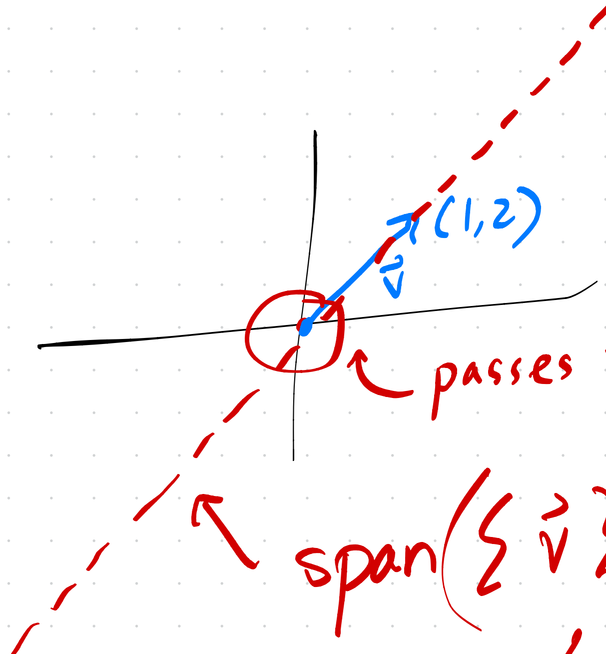
$$\operatorname{span}\left(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\}\right) = \underbrace{\left\{a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_d\vec{v}_d \mid a_1, a_2, \ldots, a_d \in \mathbb{R}\right\}}_{\text{all possible linear combos}}$$

1 vector

$\mathbb{R}^2$:

2 dimensional example

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

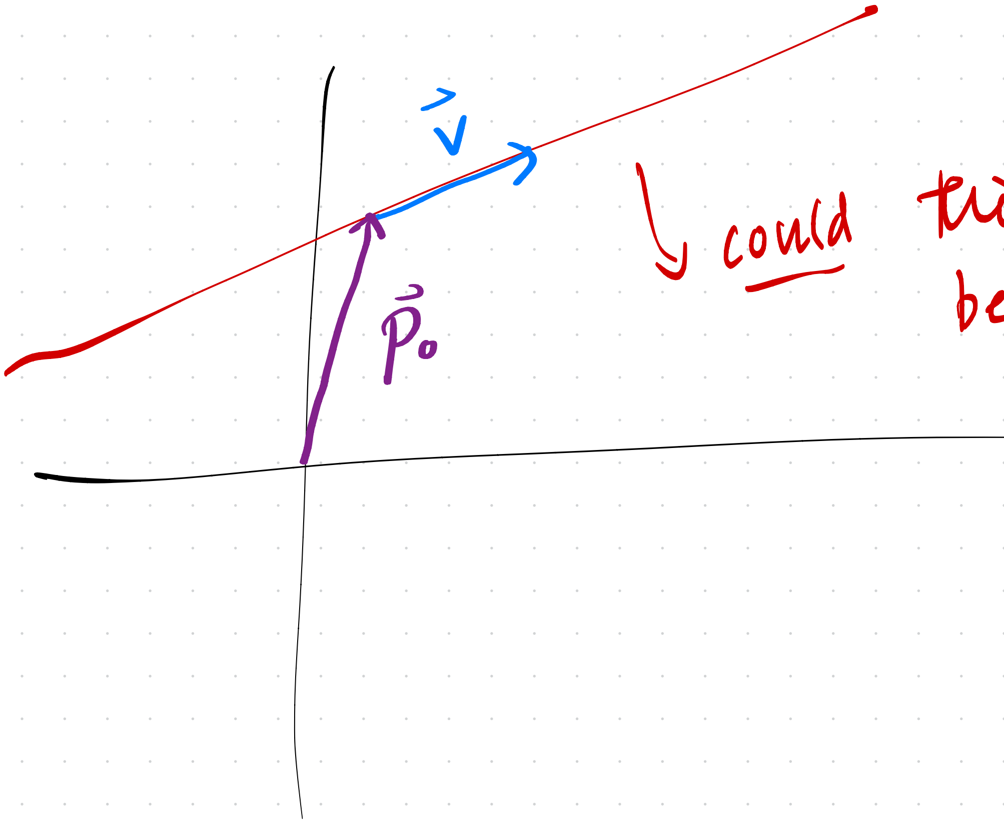


passes through (0,0)

$$\operatorname{span}(\{\vec{v}\}) = \operatorname{span}\left(\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}\right)$$

$$= \left\{ a\begin{bmatrix} 1 \\ 2 \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

$\mathbb{R}^3$: see notes for visual



could this line be the span of a vector in $\mathbb{R}^2$?

no

doesn't pass through $(0,0)$

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -7 \\ 15 \end{bmatrix}$$

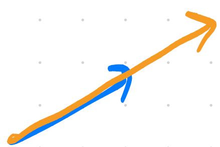
$$\mathrm{span}(\{\vec{v}\}) = \left\{ a \begin{bmatrix} 1 \\ 2 \\ 3 \\ -7 \\ 15 \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

a line in $\mathbb{R}^5$!

2 vectors

$\mathbb{R}^2$

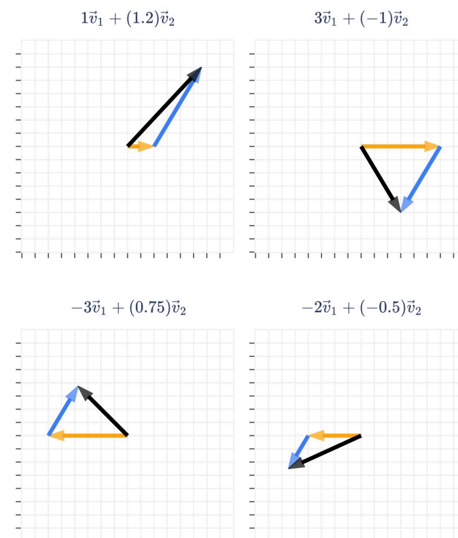
① both are $\vec{0}$

② 

collinear:
span a line

③ two vectors are not collinear

→ span all of $\mathbb{R}^2$!



$\mathbb{R}^5$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -4 \\ 15 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 3 \\ 7 \\ -1 \\ 0 \\ \pi \end{bmatrix}$

$$\underbrace{\operatorname{span}(\{\vec{v}_1, \vec{v}_2\}) = \left\{ a_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ -4 \\ 15 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ 7 \\ -1 \\ 0 \\ \pi \end{bmatrix} \;\middle|\; a_1, a_2 \in \mathbb{R} \right\}}_{\text{2-d ``subspace'' of } \mathbb{R}^5}$$

"slice"

$\underbrace{\text{3 vectors}}$

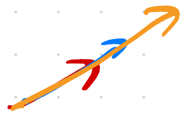
$\mathbb{R}^2$

The blue and orange vectors are redundant.
Remove either one of them,
and the remaining 2 vectors
will still span all of R^2.

Any 2 of these 3 vectors
span all of R^2.
One is redundant.

3 vectors in $\mathbb{R}^3$ **could** span:

① $\vec{0}$

② line  1d subspace of $\mathbb{R}^3$

③ plane — 2d subspace of $\mathbb{R}^3$

④ all of $\mathbb{R}^3$ — 3d subspace of $\mathbb{R}^3$ (technically)

In general,

$\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_d \in \mathbb{R}^n$ span

\# of vectors

a $\underline{\text{between } 0 \text{ and } \min(n, d)}$ -dimensional subspace

of $\mathbb{R}^n$!

dimension of the universe

we did this part on the whiteboard →

Activity: Find 6 vectors in $\mathbb{R}^4$ that span a 3-dimensional subspace of $\mathbb{R}^4$.

possible solution:

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 17 \end{bmatrix}}_{\text{3 linearly independent vectors}}, \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 13 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{can make these using lin. combs. of the first 3}} \right\}$$

2 vectors in $\mathbb{R}^4$: 0, 1, or 2-dim subspace of $\mathbb{R}^4$

4 vectors in $\mathbb{R}^2$: 0, 1, or 2-dim subspace of $\mathbb{R}^2$

Linear independence

vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are linearly independent

if either of these equivalent conditions hold:

① none of the vectors are linear combinations of the others

same!

② the only solution to $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_d\vec{v}_d = \vec{0}$

is $a_1 = a_2 = \cdots = a_d = 0$

how are these related?

suppose we have 3 vectors, $\vec{v}_1, \vec{v}_2, \vec{v}_3$

and

$$\vec{v}_3 = 3\vec{v}_1 - 4\vec{v}_2$$

these are linearly dependent

then rearrange:

$$3\vec{v}_1 - 4\vec{v}_2 - \vec{v}_3 = \vec{0}$$

in lab / hw:

given $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_5$ (for example),

how do we find a subset with the same span?

iterative algorithm

```
given v_1, v_2, ..., v_d
initialize linearly independent set S = {v_1}
for i = 2 to d:
    if v_i is not a linear combination of S:
        add v_i to S
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Ch. 4.2