EECS 245 Fall 2025

Math for ML

Lecture 8: Spans and Linear Independance

→ Read: Ch. 2.4 (new!)

Agenda

open notes directly from notes.eecs245.org ; direct links broken rn

1. Span
   → In general, given d vectors in $\mathbb{R}^n$, what can we make with their linear combinations?

2. Linear independence

Ch. 2.4

"Three Questions" linear combinations

Given $\underbrace{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d}_{d \text{ vectors}} \in \underbrace{\mathbb{R}^n}_{n \text{ components each}}$, and $\vec{b} \in \mathbb{R}^n$

① Can we write $\vec{b}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d$

i.e. is there a solution for $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_d\vec{v}_d = \vec{b}$?

② Are the a_i's unique?

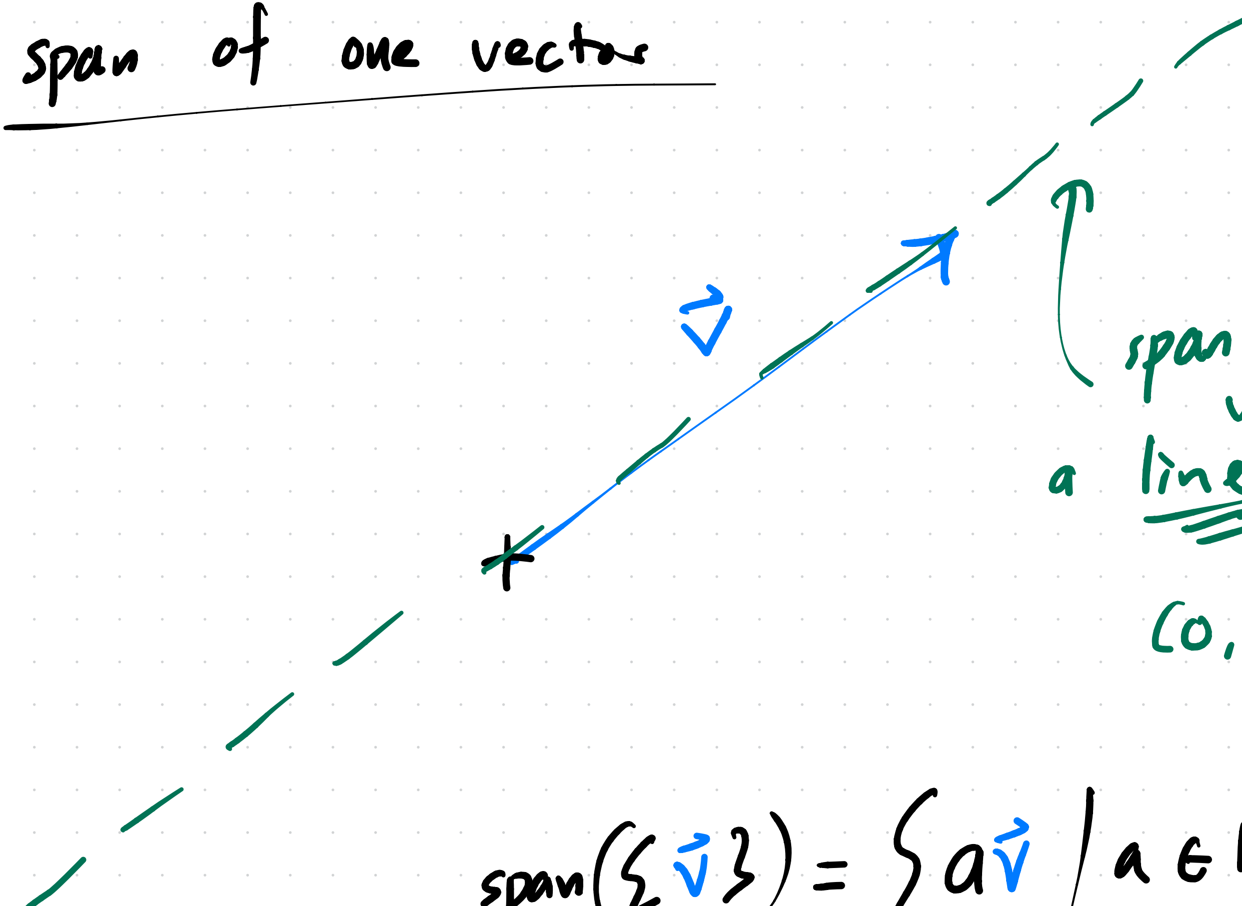
③ Span

Span

$\text{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\})$ = set of all possible linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$

$$= \{\underbrace{a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_d \vec{v}_d}_{\text{condition for inclusion}} \mid a_1, a_2, \ldots, a_d \in \mathbb{R}\}$$

span of one vector

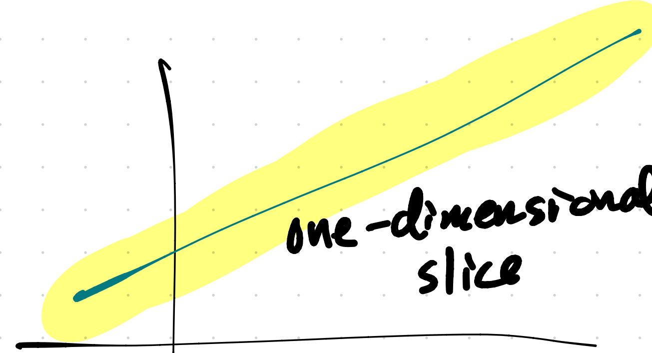


span of a single vector is a line through origin, $(0, 0, \ldots, 0)$

$$\text{span}(\{\vec{v}\}) = \{a\vec{v} \mid a \in \mathbb{R}\}$$

Aside: lines are 1-dimensional

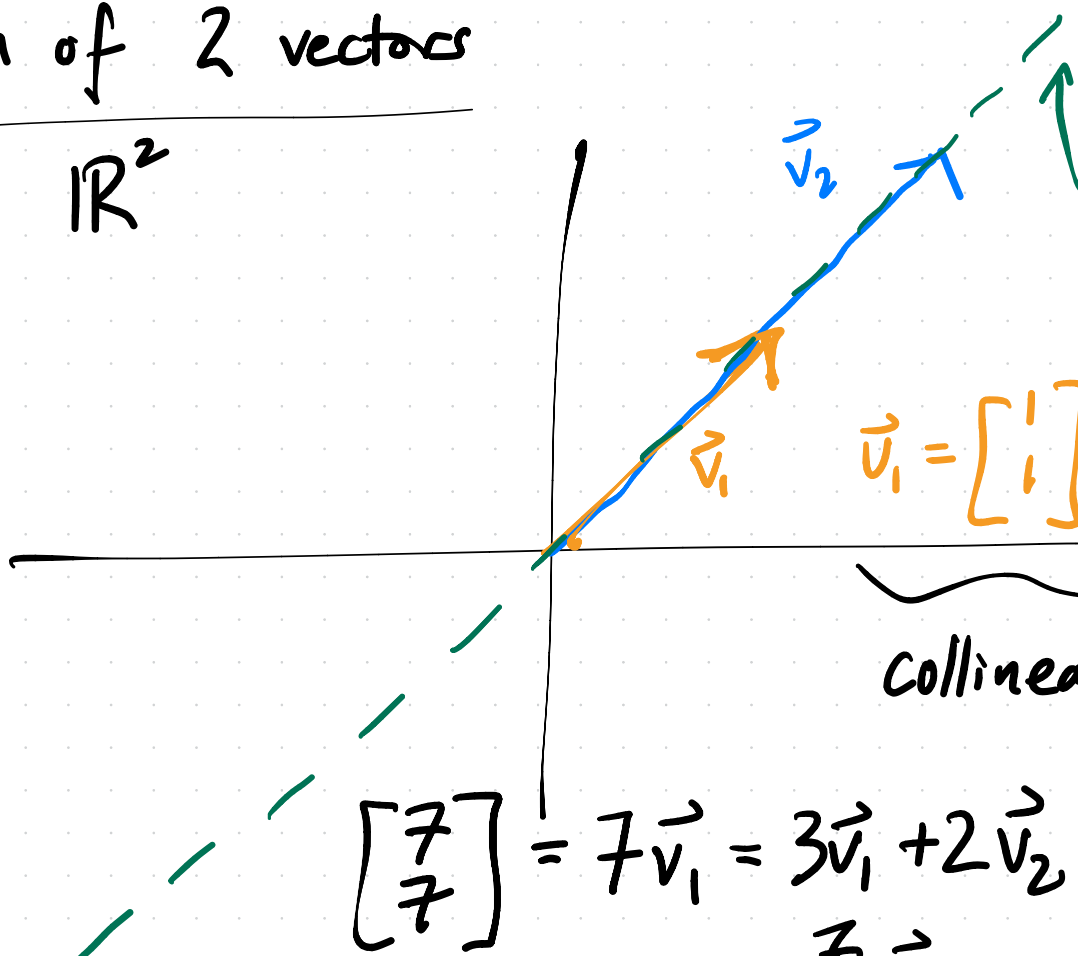
$\mathbb{R}^2$ $\qquad y = 4x - 7$



$\mathbb{R}^3, \mathbb{R}^4, \ldots,$ parametric form $\qquad L = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$

span of 2 vectors

e.x. $\mathbb{R}^2$



$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Collinear : scalar multiples of one another

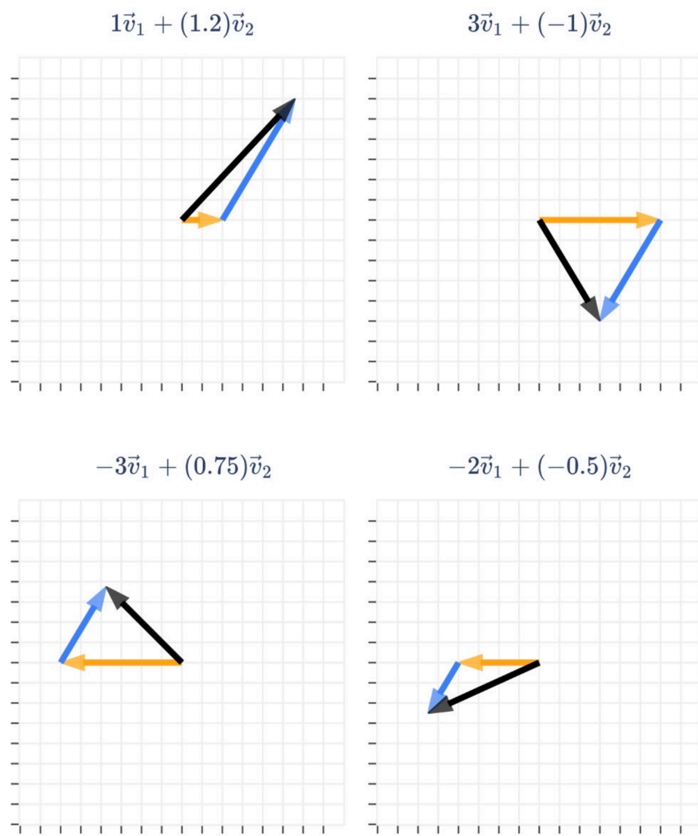
$$\begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7\vec{v}_1 = 3\vec{v}_1 + 2\vec{v}_2$$

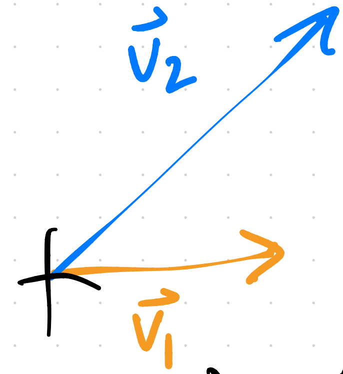
$$= \frac{7}{2}\vec{v}_2 \ldots = -35\vec{v}_1 + 21\vec{v}_2$$

$$\underbrace{\overset{\vec{v}}{\begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \\ 5 \\ 0 \end{bmatrix}} \quad \overset{3\vec{v}}{\begin{bmatrix} 3 \\ 3 \\ 6 \\ -3 \\ 15 \\ 0 \end{bmatrix}}}_{}$$

collinear: span a line in $\mathbb{R}^6$

2 vectors in $\mathbb{R}^2$, <u>not</u> colinear





$$\text{span}(\{\vec{v}_1, \vec{v}_2\}) = \mathbb{R}^2$$

for any $\vec{b} \in \mathbb{R}^2$,

solution exists

$$a_1\vec{v}_1 + a_2\vec{v}_2 = \vec{b}$$

unique, too!

2 vectors in $\mathbb{R}^3$, as long as $\vec{v}_1, \vec{v}_2$ not scalar multiples of each other,

$$\text{span}(\{\vec{v}_1, \vec{v}_2\}) = \text{plane}$$

$= $ 2-dimensional "slice" of $\mathbb{R}^3$

"standard" xy-plane

$$\text{any point} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}$$

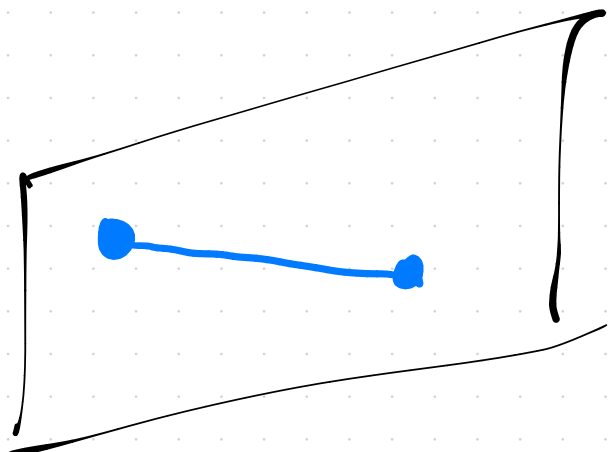
"new" coordinate system

$$\vec{v}_1 = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

any point on the plane that $\vec{v}_1$ and $\vec{v}_2$ span $= a_1 \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$

think of (a_1, a_2) "new" coordinates for the plane

plane: pick any two points on the plane.
the line connecting them
is entirely on the plane



(more to come in Ch. 2.5)

2 vectors in $\mathbb{R}^n$?

2 (non-collinear) vectors in $\mathbb{R}^n$ span a 2-dimensional **subspace** of $\mathbb{R}^n$

flat object that passes through (0,0,...) and contains all linear combinations of some set of vectors

"subspace" = "slice"

$$\vec{v}_1 = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ -8 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

"slice"

span is 2-dimensional subspace of $\mathbb{R}^5$

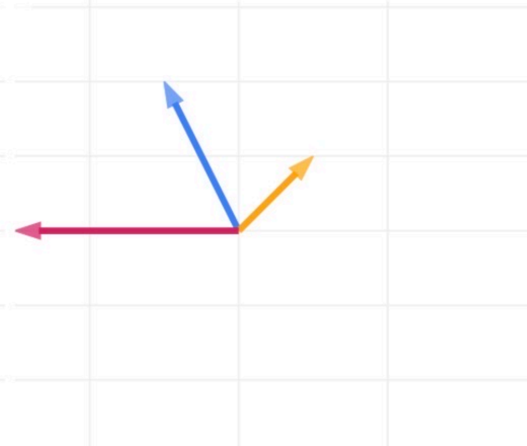
any point on $\text{span}(\vec{v}_1, \vec{v}_2)$ $= a_1 \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ -8 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

new coordinate system

3 vectors in $\mathbb{R}^2$

The blue and orange vectors are redundant.
Remove either one of them,
and the remaining 2 vectors
will still span all of R^2.

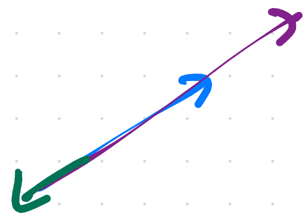
Any 2 of these 3 vectors
span all of R^2.
One is redundant.



3 vectors in $\mathbb{R}^3$

possibilities

- line



- plane

Activity

$\vec{v}_2 = -2\vec{v}_1 - \vec{v}_3$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Given that $\vec{b} = 2\vec{v}_1 + 3\vec{v}_2 - 4\vec{v}_3 + 2\vec{v}_2 - 2\vec{v}_2$,

write $\vec{b}$ as a linear combination where coefficient on $\vec{v}_2$ is 5.

Activity

6 vectors in $\mathbb{R}^4$ that span a 3d subspace of $\mathbb{R}^4$

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\3\end{bmatrix}, \begin{bmatrix}0\\0\\0\\100\end{bmatrix}$$

Linear independence

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are linearly independent

if none of the vectors are a linear combination of other vectors in the set

otherwise, they are linearly dependent

1) alternative definition

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ linearly independent

if the only way to create $\vec{0}$ as a linear combination is

$$0\vec{v}_1 + 0\vec{v}_2 + \ldots + 0\vec{v}_d .$$

e.g.

If $\quad \vec{v}_1 = \frac{2}{3}\vec{v}_2 - 7\vec{v}_3$

$$\vec{v}_1 - \frac{2}{3}\vec{v}_2 + 7\vec{v}_3 = \vec{0}$$

linearly dependent