# EECS 245 Fall 2025
# Math for ML

Lecture 9: Vector Spaces, Subspaces, and Bases

→ Read: rest of 2.4, 2.6 (new)
consult 2.5

# Midterm 1

- Tuesday, Sept. 30 in lecture (3-4:20 PM, 1013 DOW)

arrive early!

- Content:
  - Lectures 1-9 (today)
  - Chapter 1, 2.1-2.6
  - Labs 1-5
  - Homeworks 1-4

must write on paper: no iPad, no screenshots

- Allowed 1 double-sided handwritten notes sheet
- Mock exam this Friday, 2:30 - 3:50 PM + review 4-5:30PM (1365 LCSIB)

# Agenda

Big idea: formalizing what we learned last time on subspaces, dimension

1. Recap: Linear independence } 2.4 (new examples!)
2. Vector spaces and subspaces
3. "Dimension" and "basis" of a subspace } 2.6

2.5 is on lines and planes (do the activities!)

2.6 is on vector spaces, subspaces, dimension, basis (will add more details today)

# "Linear independence"

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d \in \mathbb{R}^n$ are linearly independent if

① no vector is a linear combination of any other vector

$$\vec{v}_3 = 2\vec{v}_1 + 4\vec{v}_{17}$$
$$\Downarrow$$
$$2\vec{v}_1 - 4\vec{v}_3 + 4\vec{v}_{17} = \vec{0}$$

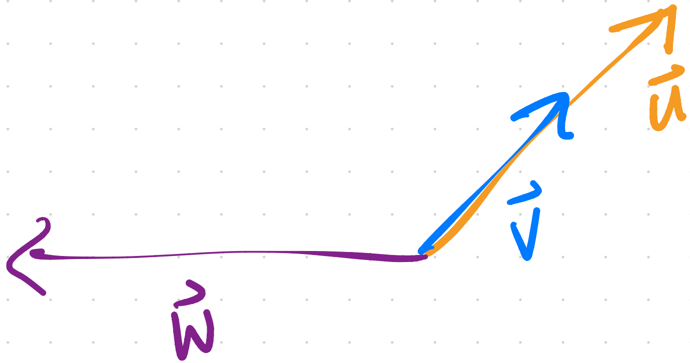
or, equivalently,

② the only solution to

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_d\vec{v}_d = \vec{0}$$

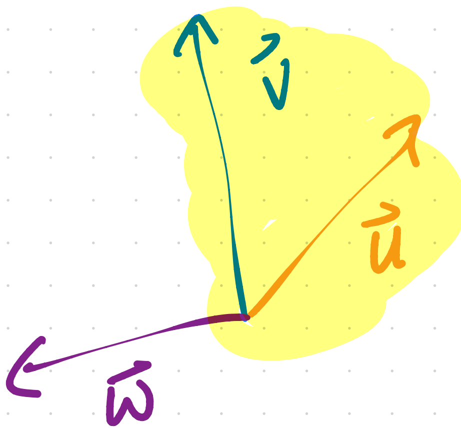
is $a_1 = a_2 = \cdots = a_d = 0$

no non-zero combination that makes $\vec{0}$




<u>not</u> linearly independent,

but still no way to make $\vec{w}$ out of $\vec{u}$, $\vec{v}$




**not** linearly independent!

any 2 of these 3 are linearly independent and **span** all of $\mathbb{R}^2$

so the third can be written as a linear combination of the others not LI ✗

LI ✓

## Activity 4

To recap what we've covered in this section, answer the following questions.

1. Can any three vectors in $\mathbb{R}^2$ be linearly independent? no!
2. **Must** any two vectors in $\mathbb{R}^2$ be linearly independent? no!
3. If two vectors in $\mathbb{R}^3$ are linearly independent, what do they span? a plane (i.e. 2-d subspace of $\mathbb{R}^3$)
4. If three vectors in $\mathbb{R}^3$ are linearly independent, what do they span? all of $\mathbb{R}^3$
5. Given $d$ vectors in $\mathbb{R}^n$, what must be true about $d$ and $n$ for it to be possible for the vectors to be linearly independent? $d \le n$

**Solutions**

3:41

Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d \in \mathbb{R}^n$ are orthogonal, meaning if $i \neq j$

$$\vec{v}_i \cdot \vec{v}_j = 0,$$

and none are the zero vector!

Prove $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are linearly independent.

formal definition of linear independence want to show that the only solution to:

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_d \vec{v}_d = \vec{0}$$

is $a_1 = a_2 = \dots = a_d$

dot product with $\vec{v}_1$ on both sides

$$(a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_d \vec{v}_d) \cdot \vec{v}_1 = \vec{0} \cdot \vec{v}_1$$

$$a_1 \vec{v}_1 \cdot \vec{v}_1 + \cancel{a_2 \underbrace{\vec{v}_2 \cdot \vec{v}_1}_{\text{orthogonal!}}}^{0} + \dots + \cancel{a_d \vec{v}_d \cdot \vec{v}_1}^{0} = 0$$

$$a_1 \vec{v}_1 \cdot \vec{v}_1 = 0 \longrightarrow \underbrace{a_1}_{=0} \underbrace{\|\vec{v}_1\|^2}_{>0} = 0$$

so $a_1 = 0$

repeat for $\vec{v}_2, \vec{v}_3, \dots, \vec{v}_d$

2.6

addition
scalar multiplication $\Big\}$ linear combinations

A vector space, $V$, is a set of objects where:

1. if $\vec{u} \in V$, $\vec{v} \in V$, $\vec{u}+\vec{v} \in V$
2. if $c \in \mathbb{R}$, $\vec{v} \in V$, $c\vec{v} \in V$

e.g.

$V =$ the set of polynomials of degree $\leq 3$

$$u(x) = 2x^2 - x + 4$$

$$v(x) = 5x^3 + 2x^2 - 7x + 3$$

need $\leq 3$

$$2x^3 + 5$$

$$-2x^3 + 5$$

# Subspaces

A subspace $S$ of a vector space $V$ is a set where:

any linear combination is contained in the subsp.

"closure"

1. $\vec{0} \in S$
2. if $\vec{u} \in S$, $\vec{v} \in S$, then $\vec{u} + \vec{v} \in S$
3. $\vec{v} \in S$, $c\vec{v} \in S$

a subspace is a vector space contained within another vector space e.g. line through $(0,0)$ in $\mathbb{R}^2$

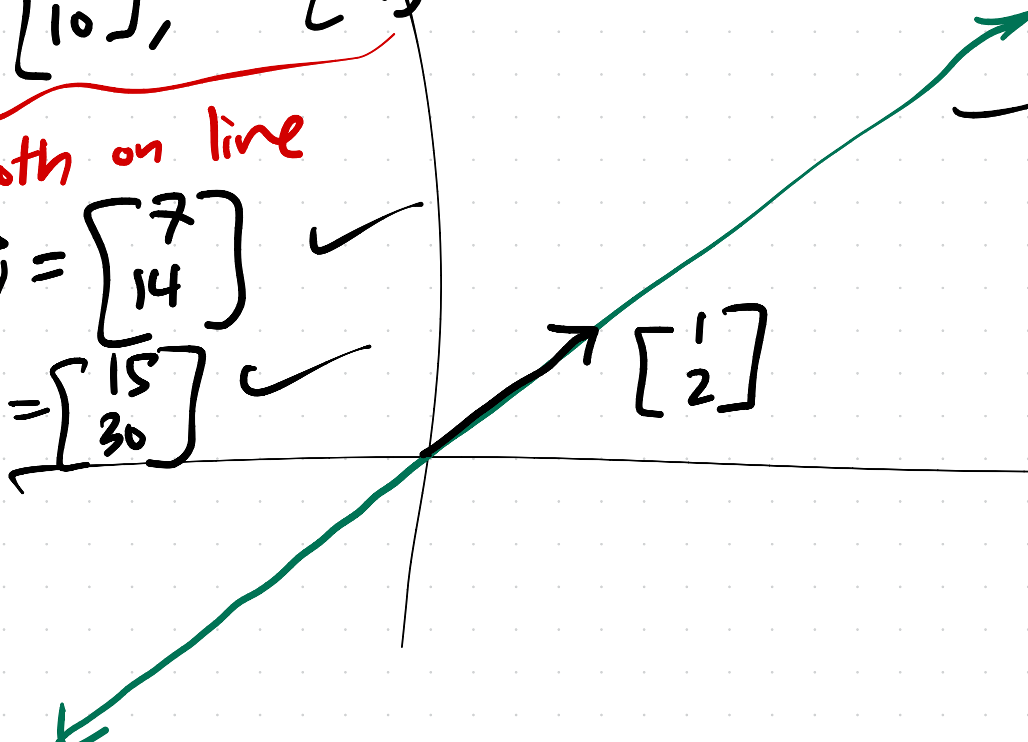
subspace of $\mathbb{R}^2$!

$\vec{u} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

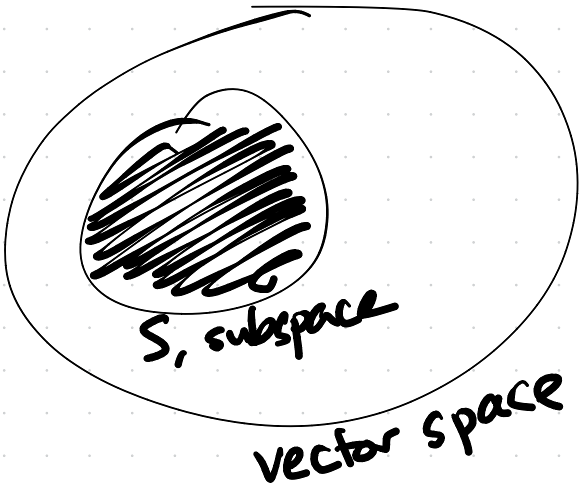
both on line

$\vec{u} + \vec{v} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$ ✓

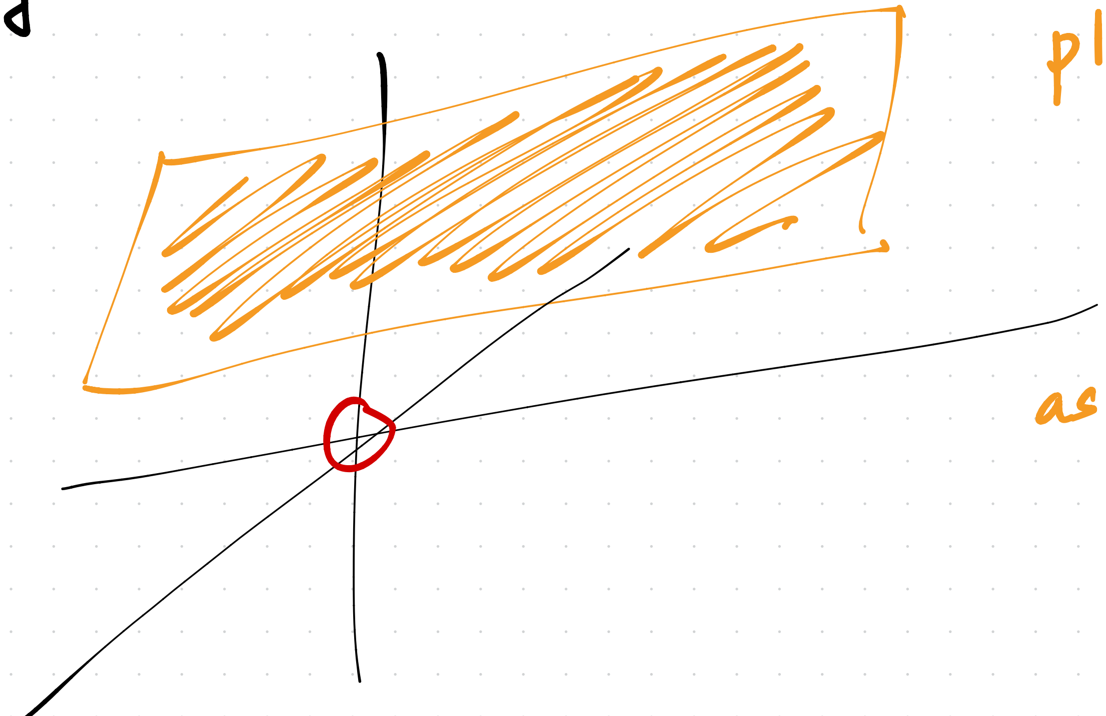
$3\vec{u} = \begin{bmatrix} 15 \\ 30 \end{bmatrix}$ ✓




e.g.
$L = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$,
$t \in \mathbb{R}$

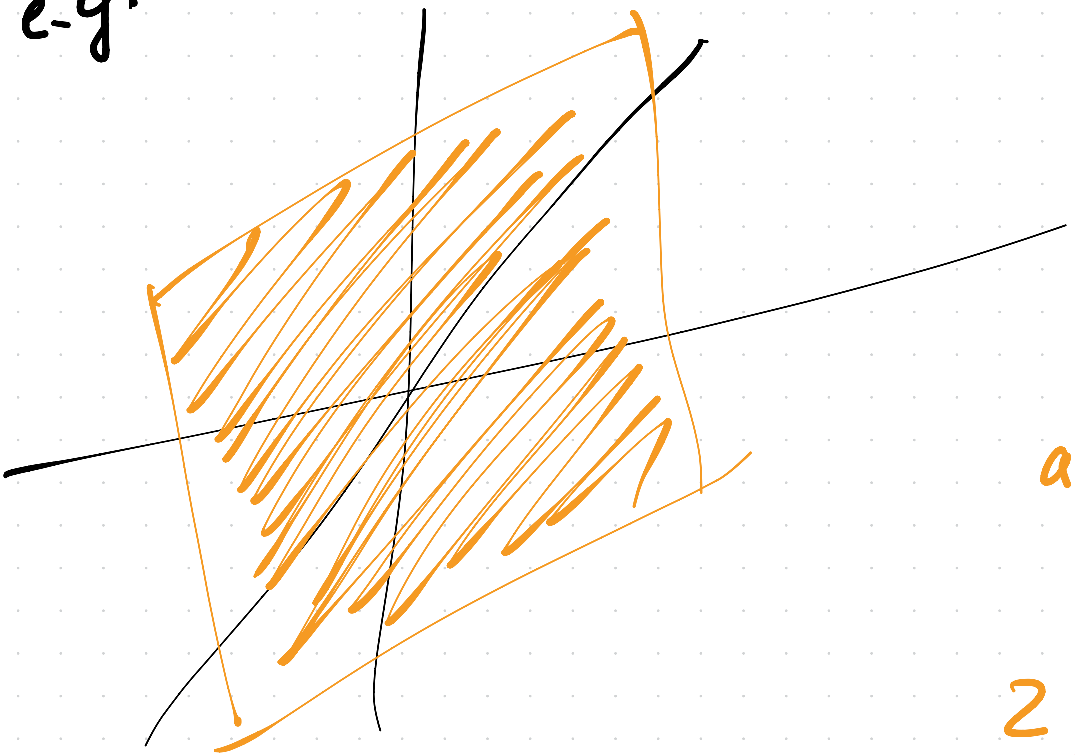

S, subspace
vector space

e.g.



this plane in $\mathbb{R}^3$ is not a subspace, as it doesn't contain $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

e.g.



plane through $(0,0,0)$ in $\mathbb{R}^3$ is a subspace, since it's the span of 2 vectors in $\mathbb{R}^3$

set of all linear combinations

$$\text{span}\left(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\}\right)$$

is

a subspace

of $\mathbb{R}^n$

Ex: set of vectors in $\mathbb{R}^5$ where $2^{nd}$ and $4^{th}$ components are equal

is this a subspace of $\mathbb{R}^5$?

$$\vec{v} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ b_1 \\ d_1 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ b_2 \\ d_2 \end{bmatrix} \qquad \vec{u}+\vec{v} = \begin{bmatrix} a_2+a_1 \\ b_2+b_1 \\ c_2+c_1 \\ b_2+b_1 \\ d_2+d_1 \end{bmatrix}$$

both still equal!

$\vec{0} \in$ set too

$$c\vec{v} = \begin{bmatrix} ca_1 \\ cb_1 \\ cc_1 \\ cb_1 \\ cd_1 \end{bmatrix}$$

1. $\vec{u}+\vec{v} \in S$
2. $c\vec{v} \in S$

equivalently

$$c\vec{u}+d\vec{v} \in S$$

set of vectors in $\mathbb{R}^2$ where first component is 1

subspace of $\mathbb{R}^2$?

no!

① doesn't contain $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

② not "closed"

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}_{\text{not in set}}$$

**Basis** for a subspace $S$:

set of vectors that

① are <u>linearly</u> <u>independent</u>

② span all of $S$

eg. $\text{span}\left\{ \overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}, \overset{\vec{v}_3}{\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}} \right\}$

plane in $\mathbb{R}^3$

basis: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

...

3 possible bases (plural of basis)

for $\text{span}\left(\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}3\\4\\5\end{bmatrix}\right\}\right)$

$\Rightarrow$ a subspace has many possible bases

$\Rightarrow$ but, all of these bases have the same number of vectors

$\Rightarrow$ that number is called the dimension of the subspace