EECS 245 Fall 2025
Math for ML

Lecture 10: Bases and Dimension ;
The "Curse of Dimensionality"

→ Read 2.6 (new examples!)

Agenda

① Recap: Subspaces, bases, dimension

all in Ch. 2.6, which now has new examples

→ in scope for Midterm 1 (also essentially on Homework 4)

② The "Curse of Dimensionality"

→ programming demo; not in scope

Vector space

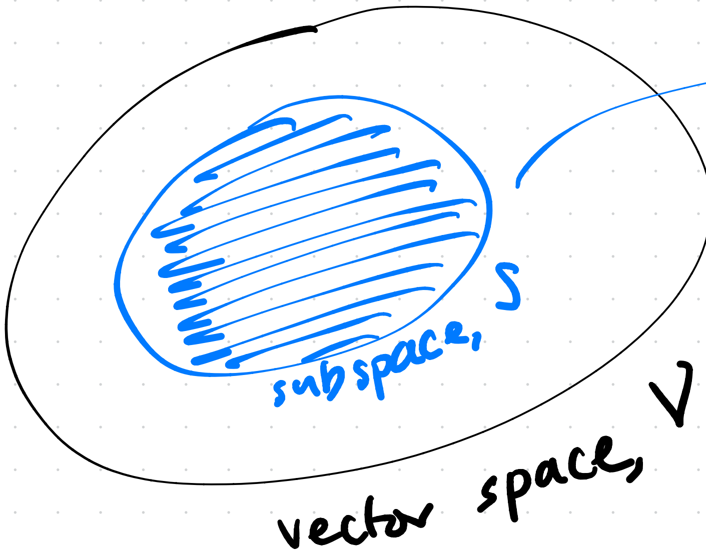
$$\vec{c} = \vec{a} + \vec{b}$$

$$\vec{e} = \vec{c} + \vec{d}$$

$$= (\vec{a} + \vec{b}) + \vec{d}$$

$$\vdots$$

can add any number of items together, not just 2



to check if S is a subspace of V

1. check if $\vec{0} \in S$
2. if $\vec{v} \in S$, then $c\vec{v} \in S$, for any c
3. if $\vec{u}, \vec{v} \in S$, then $\vec{u} + \vec{v} \in S$

if $\vec{u}, \vec{v} \in S$,
$c\vec{u} + d\vec{v} \in S$
equivalent

Example : First component is non-negative, using $\mathbb{R}^2$

in set notation,

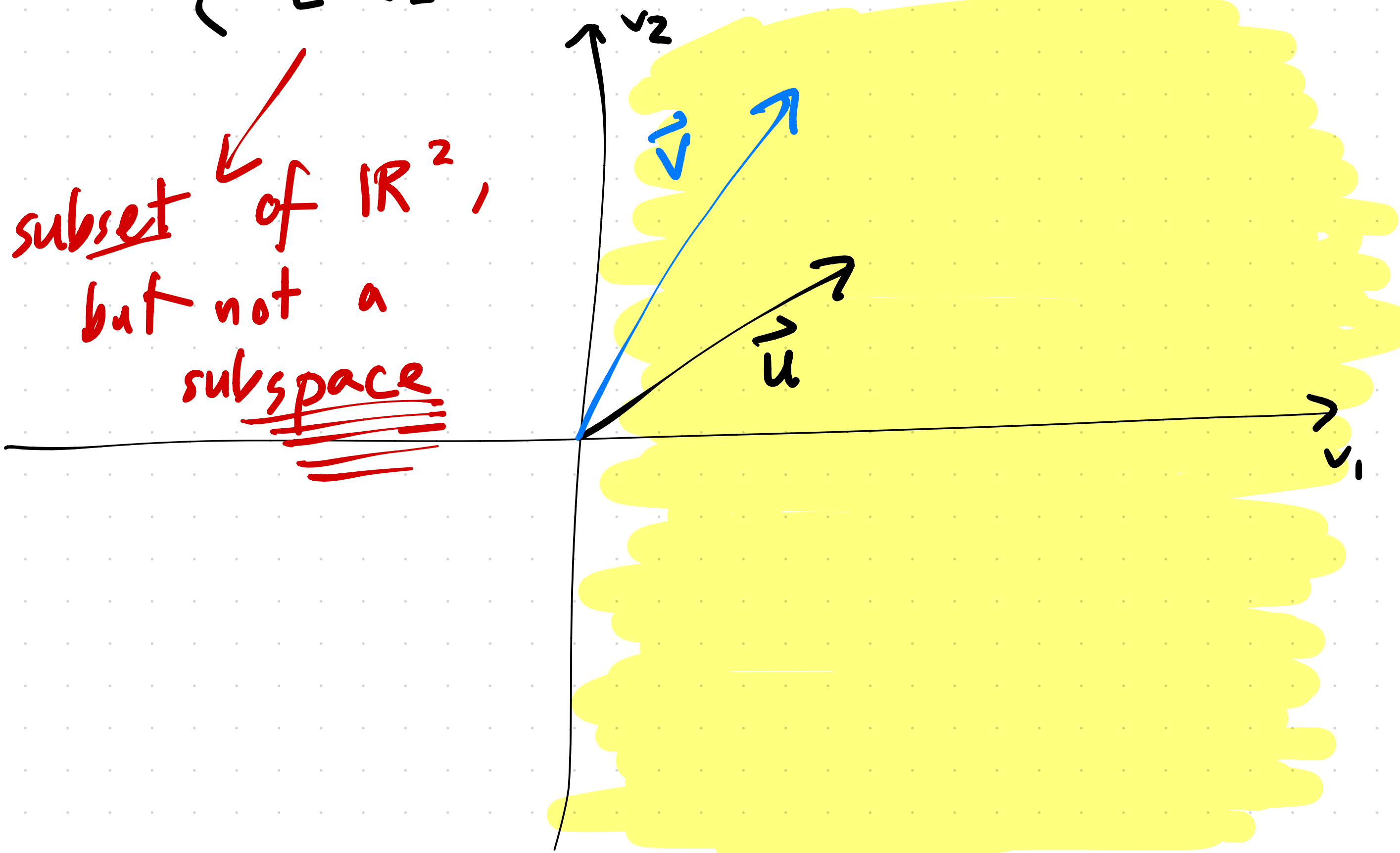
$$\left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \middle| \; v_1 \geq 0, \; v_2 \in \mathbb{R} \right\}$$

is this a subspace of $\mathbb{R}^2$? no

e.g. $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\underbrace{-\vec{u} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}}_{\text{not in set,}}$ so set can't be subspace

$$\left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \;\middle|\; v_1 \geq 0,\; v_2 \in \mathbb{R} \right\}$$

subset of $\mathbb{R}^2$, but not a subspace



$$\underbrace{\left\{ \vec{v} \in \mathbb{R}^5 \;\middle|\; \|\vec{v}\| = 1 \right\}}$$

not a subspace!
doesn't contain $\vec{0}$

$$\left\{ \vec{v} \in \mathbb{R}^5 \;\middle|\; \|\vec{v}\| \leq 1 \right\}$$

not a subspace!

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

not a unit vector

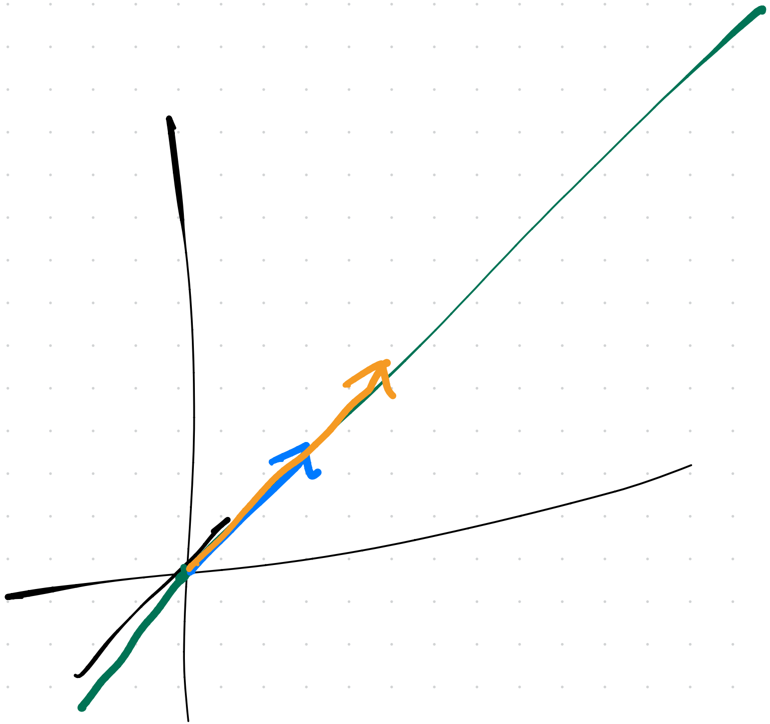
Line

$$L = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} t, \quad t \in \mathbb{R}$$

(set notation)

$$= \underbrace{\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} t \;\middle|\; t \in \mathbb{R} \right\}}_{}$$

set of all vectors that can be written in this form



Important

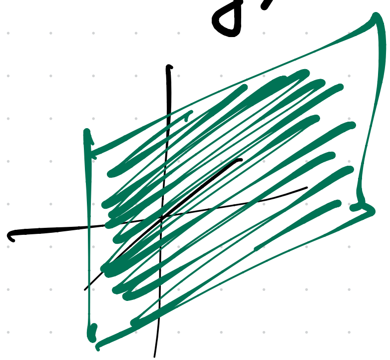
$$\text{span}(\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d\})$$

is always a subspace!

$$\vec{u} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$$

$$\operatorname{span}(\{\vec{u}, \vec{v}\}) = \{ a\vec{u} + b\vec{v} \mid a, b \in \mathbb{R} \}$$

visually, span($\{\vec{u}, \vec{v}\}$) is a plane in $\mathbb{R}^3$



pick 2 vectors in $\text{span}(\{\vec{u}, \vec{v}\})$

$$\vec{x} = 6\vec{u} + 3\vec{v}$$
$$\vec{y} = -3\vec{u} + 2\vec{v}$$

take a linear combination of $\vec{x}, \vec{y}$

$$3\vec{x} - 4\vec{y} = 3(6\vec{u}+3\vec{v}) - 4(-3\vec{u}+2\vec{v})$$

$$= \underbrace{30\vec{u} + \vec{v}}_{\text{a linear combination of } \vec{u}, \vec{v},\ \text{so, it must be in span!}}$$

example

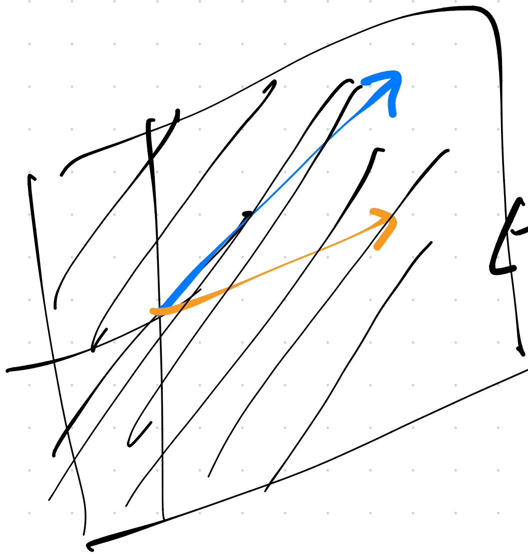
$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; \overbrace{x+y+z=0}^{\text{plane through } (0,0,0)} \right\}$$

is this a subspace? yes!

it's a plane in $\mathbb{R}^3$ through $(0,0,0)$, so it's the span of 2 vectors in $\mathbb{R}^3$

e.g. $\vec{u} = \begin{bmatrix} 30 \\ 40 \\ -70 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -5 \\ 10 \\ 5 \end{bmatrix}$

$$\vec{u} = \begin{bmatrix} 30 \\ 40 \\ -70 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -5 \\ 10 \\ 5 \end{bmatrix}$$



$x + y + z = 0$

spans are subspaces
AND
every subspace is spanned by some vectors

$x + y + z = 5$

not a subspace! $(0,0,0)$ not in it

$$\operatorname{span}\left(\left\{\begin{bmatrix}1\\5\\6\end{bmatrix}, \begin{bmatrix}0\\-1\\2\end{bmatrix}\right\}\right)$$

is a subspace!
every span is a subspace

read 2.5 to find the plane of the form

$$ax+by+cz=0$$

that these 2 span

break till 3:51

standard form

$$2x+3y-7z=0$$

parametric form

$$\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}-3\\2\\0\end{bmatrix}$$

$$P = s\begin{bmatrix}2\\1\\1\end{bmatrix} + t\begin{bmatrix}-3\\2\\0\end{bmatrix}, \quad s,t \in \mathbb{R}$$

Basis

for a subspace S is a set of vectors that

① are linearly independent

② span all of S

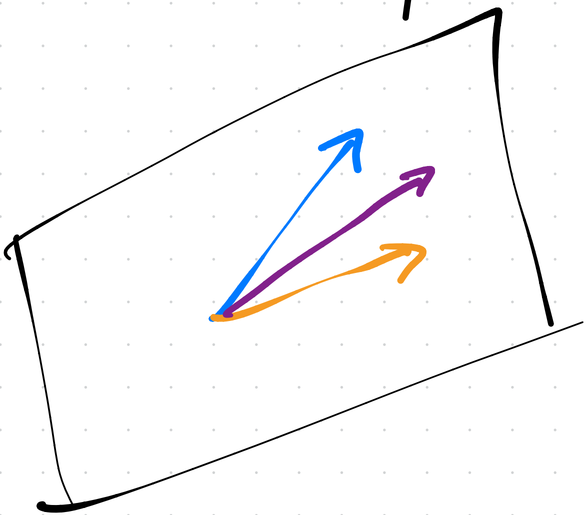
$\vec{v}_3 = \vec{v}_1 + \vec{v}_2$

e.g. $\text{span}\left(\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}3\\5\\7\end{bmatrix}\right\}\right)$

all lie on <u>same</u> plane

$$S = \text{span}\left(\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}3\\5\\7\end{bmatrix}\right\}\right) \qquad \dim(S) = 2$$

is a plane in $\mathbb{R}^3$ (2 dimensional)



basis for this subspace:

$$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}\right\},$$

$$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\5\\7\end{bmatrix}\right\}$$

"Standard basis" for $\mathbb{R}^2$

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\dim(\mathbb{R}^2) = 2$

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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another basis for $\mathbb{R}^2$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} -5 \\ 17 \end{bmatrix}$ $\rightarrow$ also a basis, the numbers are just uglier lol

Dimension of a subspace S

is the number of vectors in a basis for S

$\dim(S)$

every basis of S must have the same number of vectors, i.e. dimension

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ basis for $\mathbb{R}^3$? no

doesn't span all

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}3\\4\\5\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ basis for $\mathbb{R}^3$ yes

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}3\\4\\5\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ basis for $\mathbb{R}^3$? no

$a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}3\\4\\5\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$

aren't linearly independent

$$\underbrace{\left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix} \right\}}$$

on it's own
not a subspace

but, span( )
is a subspace

① Find 2 possible bases for (subspace of $\mathbb{R}^3$)

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; \underbrace{2x-3y+4z=0}_{\text{plane through } (0,0,0)} \right\}$$

What is its $\underbrace{\text{dimension}}_{\dim = 2}$?

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} \right\}, \quad \left\{ \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \right\}$$

Problem 4: Finding a Linearly Independent Subset (7 pts)

basis

In each of the parts below, using the algorithm mentioned in this section of Chapter 2.4, find **a** linearly independent set of vectors that spans the same span as the given set of vectors.

In your solutions, show all of the steps of the algorithm, clearly state what the vectors in the linearly independent set are, and how many vectors are in the set.

dimension

There are multiple possible answers for each part, but all of them have the same number of vectors.

a) (3 pts)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

b) (4 pts)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Problem 5: Rows and Columns (12 pts)

Soon, we will start to learn about matrices. In this problem, we'll start to connect what we've learned about vectors and spans to matrices.