EECS 245 Fall 2025

Math for ML

Lecture 11: Matrices

→ Read: Ch. 2.7

About exam grades...

my freshman transcript

Fall 2016

Class	Title	Un.	Gr.
CHEM 1A	General Chemistry	3	B-
CHEM 1AL	General Chemistry Laboratory	1	C+
COMPSCI 61A	The Structure and Interpretation of Computer Programs	4	B+
COMPSCI 70	Discrete Mathematics and Probability Theory	4	A
COMPSCI 195	Social Implications of Computer Technology	1	P
MATH 1A	Calculus	4	A+

Spring 2017

Class	Title	Un.	Gr.
COMPSCI 61B	Data Structures	4	B+
COMPSCI 97	Field Study	1	P
COMPSCI 197	Field Study	1	P
ELENG 16A	Designing Information Devices and Systems I	4	B-
MATH 110	Linear Algebra	4	C
MATH 128A	Numerical Analysis	4	B+

EECS 245-ish

Math 217

.... but still, grades <u>do</u> matter,

and the easiest path to a good grade is to

actually do the labs and homeworks yourself

WITHOUT ChatGPT

otherwise, you're just cheating yourself!

exams 70%, homeworks 30%

Agenda

1. Matrices: definition, addition, scalar multiplication
2. Matrix-vector multiplication
3. Matrix-matrix multiplication
4. Transpose and identity
5. (If time, otherwise Tuesday) rank, CR decomposition

HW 5 out tomorrow!

Matrix: Rectangular grid of numbers

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 9 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^4$$

$$A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & 9 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

4×3

A has 4 rows, 3 columns, $A \in \mathbb{R}^{4 \times 3}$

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & 9 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

$$A_{31}, \quad A_{3,1}, \quad A_{i,j}$$

element in row 3, column 1

In general,

$$A \in \mathbb{R}^{n \times d}$$

\# of columns

$$\begin{bmatrix} 1 & 1 \\ 3 & 4 \\ -2 & 0 \\ 5 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 3 & 1 & 0 \\ 1 & 0 & 5 & 1 & 5 \\ 3 & 0 & 8 & 1 & 3 \end{bmatrix}$$

"tall"	"square"	"wide"
$n > d$	$n = d$	$n < d$

matrices support addition and scalar multiplication

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & 9 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

$$4A - 5B = \begin{bmatrix} 7 & & \\ & & \\ & -44 & \\ & \underline{\quad} & \end{bmatrix}$$

"Golden rule" of matrix multiplication

Suppose A, B matrices.

In order for the product

$$AB$$

to exist,

columns in A = # rows in B

Matrix-vector

think of a vector is just a matrix with 1 column

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & 9 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}_{4\times 3} \qquad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}_{3 \times 1}$$

"inner dimensions must match"

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & 9 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

key idea: $A\vec{x}$ will be a new vector containing the $\boxed{\text{dot product}}$ of $\vec{x}$ with every <u>row</u> of A!

$$\underbrace{4 \times 3}_{A} \quad \underbrace{3 \times 1}_{\vec{x}} = \underbrace{4 \times 1}_{\text{output}}$$

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & 9 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$(3)(1) + (1)(0) + (4)(3)$$

$$A\vec{x} = \begin{bmatrix} 15 \\ 29 \\ 0 \\ 2 \end{bmatrix}_{4\times 1}$$

"computational" interpretation

conceptual

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & 9 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$A\vec{x} = \underbrace{1\begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ -1 \\ -2 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 9 \\ 0 \\ 0 \end{bmatrix}}_{\text{linear combination of columns of } A!}$$

$A\vec{x}$ is a linear combination of the columns in A, using the coefficients in $\vec{x}$

$$M = \begin{bmatrix} 2 & -1 & 3 & 0 & 4 \\ 1 & 5 & -2 & 1 & 0 \end{bmatrix}$$

$\vec{u} \in \mathbb{R}^5$

each description is of a new vector $\vec{u}$:

1) a vector whose second component is 1, rest are 0

2) a vector with all components $= 1/5$

3) a vector with first component $= 3/5$ $\rightarrow \vec{u} = \begin{bmatrix} 3/5 \\ 1/10 \\ 1/10 \\ 1/10 \\ 1/10 \end{bmatrix}$

all components sum to 1

weighted mean of columns

$$M = \begin{bmatrix} 2 & -1 & 3 & 0 & 4 \\ 1 & 5 & -2 & 1 & 0 \end{bmatrix}$$

1) $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\qquad M\vec{u} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

2) $\vec{u} = \begin{bmatrix} 1/5 \\ 1/5 \\ 1/5 \\ 1/5 \\ 1/5 \end{bmatrix}$ $\qquad M\vec{u} = \begin{bmatrix} \bigcirc \\ \\ \end{bmatrix}$ = average of the columns!

mean of row 1:

$2\left(\frac{1}{5}\right) + (-1)\left(1/5\right) + \ldots + (4)\left(1/5\right)$

Matrix-matrix multiplication

$\vec{x}$ before

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & 9 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 7 \\ 3 & 2 \end{bmatrix}$$

4×3 $\qquad$ 3×2

$$AB = \begin{bmatrix} 15 & 21 \\ 29 & 29 \\ 0 & -7 \\ 2 & -10 \end{bmatrix} \; 4 \times 2$$

element $(3, 2)$
dot product of
row 3 in A and
col 2 of B

In general,

$$A \in \mathbb{R}^{n \times d} \qquad B \in \mathbb{R}^{d \times p}$$

then

$$AB \in \mathbb{R}^{n \times p} \quad \text{where}$$

$$(AB)_{ij} = \begin{pmatrix} \text{row } i \\ \text{in } A \end{pmatrix} \cdot \begin{pmatrix} \text{column } j \\ \text{in } B \end{pmatrix}$$

row i, column j

properties

1) $(AB)C = A(BC)$ ✓

as long as shapes satisfy "Golden Rule"

2) $A(B+C) = AB + AC$ ✓

3) $AB \neq BA$

each is made up of different dot products

$A_{5 \times 3}$ $\quad$ $B_{3 \times 6}$ $\quad$ AB exists,

but $B_{3 \times 6} A_{5 \times 3}$ doesn't

don't match!

$A_{3\times 6}$ $B_{6\times 3}$

$$AB \in \mathbb{R}^{3\times 3}$$

$$BA \in \mathbb{R}^{6\times 6}$$

both exist, but have different dimensions

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

even if AB, BA same shape, aren't equal in general!

$$BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

row 2 of B · column 2 of A

1) Transpose of matrix

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & 9 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}_{4\times 3}$$

transpose

$$A^T = \begin{bmatrix} 3 & 2 & 0 & 2 \\ 1 & 1 & -1 & -2 \\ 4 & 9 & 0 & 0 \end{bmatrix}$$

row 2 → column 2

3x4

rows → columns

columns → rows

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\vec{u}^T \vec{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$= \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

important :

$$(AB)^T = B^T A^T$$

application :

$$\|\underbrace{A\vec{x}}_{\text{vector}}\|^2 = (A\vec{x}) \cdot (A\vec{x}) = (A\vec{x})^T (A\vec{x})$$

$$= \vec{x}^T A^T A \vec{x}$$