

EECS 245, Winter 2026

LEC 12 Rank, Column Space, and Null Space

→ Read Ch. 5.3-5.4

# Agenda

- Recap: Matrices, transpose
- Four big ideas:
  - Column space
  - Rank
  - Null space
  - Rank-Nullity theorem

# Announcements

- HW 6 due on Friday
- MT 1 regrades due tomorrow
- Sign up for a 1-on-1 check-in (email me if none of the slots work for you)
- Lab this week!

$$\begin{aligned}\|A\vec{x}\|^2 &= (A\vec{x}) \cdot (A\vec{x}) \\ &= (A\vec{x})^T (A\vec{x}) \\ &= \vec{x}^T A^T A \vec{x}\end{aligned}$$

$A$  :  $n \times d$  matrix  
↑ rows  
↖ cols

$$\begin{aligned}(AB)^T &= B^T A^T \\ AB &\neq BA\end{aligned}$$

$\vec{x}$  :  $\mathbb{R}^d$  (  $d \times 1$  matrix )  
vector

$A\vec{x}$  :  $\mathbb{R}^n$  (  $n \times 1$  matrix )  
vector

# Aside: Transpose

$$A_{n \times d} \rightarrow A^T_{d \times n}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \vec{v}^T \vec{u} = \vec{v} \cdot \vec{u}$$

(not  ~~$\vec{u}^T \cdot \vec{v}$~~ )

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & -1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$$A\vec{x} = (2) \begin{bmatrix} 5 \\ 0 \\ 3 \\ 6 \\ -1 \end{bmatrix} + (0) \begin{bmatrix} 3 \\ -1 \\ 4 \\ 2 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ -1 \\ -1 \\ 4 \\ -1 \end{bmatrix} \in \mathbb{R}^5$$

$A\vec{x}$  is a lin. comb. of  $A$ 's columns!

column space of matrix  $A$  is

- the set of all linear combinations of  $A$ 's columns
- i.e. span of  $A$ 's columns
- set of outputs of  $A\vec{x}$ , for all possible  $\vec{x}$ 's

each column is in  $\mathbb{R}^n$

$$A_{n \times d} = \begin{bmatrix} | & | & \dots & | \\ \vec{a}^{(1)} & \vec{a}^{(2)} & \dots & \vec{a}^{(d)} \\ | & | & \dots & | \end{bmatrix}$$

$$\text{colsp}(A) = \text{span} \left( \underbrace{\left\{ \vec{a}^{(1)}, \vec{a}^{(2)}, \dots, \vec{a}^{(d)} \right\}}_{\text{subspace of } \mathbb{R}^n} \right)$$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & -1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\dim(\text{colsp}(A)) = 2$$

important:

$$\text{col } 1 = \text{col } 2 + \text{col } 3$$

$$\text{colsp}(A) = \text{span} \left( \left\{ \begin{bmatrix} 5 \\ 0 \\ 3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \\ 4 \\ -1 \end{bmatrix} \right\} \right)$$

not linearly independent!

$$= \text{span} \left( \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \\ 4 \\ -1 \end{bmatrix} \right\} \right) = \text{span}(\{\text{col } 2, \text{col } 3\})$$

$\text{rank}(A) =$  dimension of  
column space of  $A$

$=$  # linearly independent  
columns in  $A$

↑  
don't ever forget this!

①  $3 \times 4$  matrix of rank 1  
rows cols

colsp is a line in  $\mathbb{R}^3$

$$\begin{bmatrix} 1 & 2 & -1 & 10 \\ 1 & 2 & -1 & 10 \\ 2 & 4 & -2 & 20 \end{bmatrix}$$

"

"

rank 2

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

rank 3

$$\begin{bmatrix} 2 & 0 & 0 & 8 \\ 0 & 8 & 0 & 12 \\ 0 & 0 & 5 & 34 \end{bmatrix}$$

can't make  $3 \times 4$  matrix with rank 4!

not all pairs of columns span the colsp, but colsp is spanned by just 2 cols



$$\vec{u}^T \vec{v} = 2 - 15 - 4 = -17 = \vec{u} \cdot \vec{v} \rightarrow \text{dot product / inner product}$$

cols multiples of  $\vec{u}$ ,  
rows multiples of  $\vec{v}^T$ !

$$\vec{u} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$$

"outer product" of  $\vec{u}, \vec{v}$   $\leftarrow$  vectors

$$\vec{u}_{3 \times 1} \vec{v}_{1 \times 3}^T$$

$$= \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

$$[2 \ 5 \ -1]$$

$$= \begin{bmatrix} 2 & 5 & -1 \\ -6 & -15 & 3 \\ 8 & 20 & -4 \end{bmatrix}$$

$\vec{v}^T$   
 $-3\vec{v}^T$   
 $4\vec{v}^T$   
 $3 \times 3$

$2\vec{u}$      $5\vec{u}$      $-\vec{u}$

$$\text{rank}(\vec{u}\vec{v}^T) = 1$$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 5 & 0 & 3 & 6 & 1 \\ 3 & -1 & 4 & 2 & 0 \\ 2 & 1 & -1 & 4 & 1 \end{bmatrix}$$

$\text{rowsp}(A) = \text{span of } A\text{'s rows!}$

$$= \text{colsp}(A^T)$$

row space is the set of all possible outputs of  $A^T \vec{y}$ ,  $\vec{y} \in \mathbb{R}^5$

"row space"

=

colsp ( $A^T$ )

$$A^T = \begin{bmatrix} 5 & 0 & 3 & 6 & 1 \\ 3 & -1 & 4 & 2 & 0 \\ 2 & 1 & -1 & 4 & 1 \end{bmatrix}$$

$\vec{a}_2$                        $\vec{a}_5$

$\text{colsp}(A^T)$  is a 2-dimensional subspace of  $\mathbb{R}^3$   
 $\vec{a}_2, \vec{a}_5$  can span other cols of  $A^T$

$$\text{rank}(A^T) = \underline{2} = \text{rank}(A)$$

Fact : For any matrix  $A$ ,

# of linearly independent columns

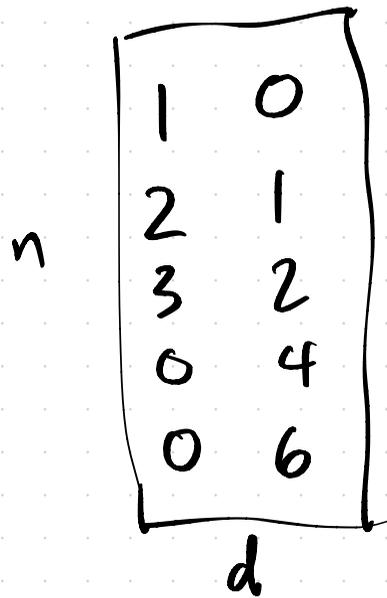
# of linearly independent rows

$$\text{rank}(A) = \text{rank}(A^T)$$

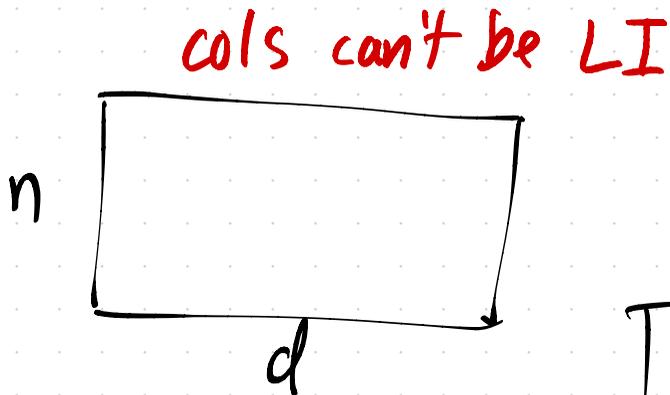
i.e.

$A$  is an  $n \times d$  matrix  
rows can't be LI

$$\text{rank}(A) = \text{rank}(A^T)$$



$$n > d$$



$$n < d$$

either both  
the cols & rows  
are LI,  
or neither

$$n = d$$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

since  $A\vec{x} = \vec{0}$ ,  
 $\vec{x}$  is in the  
NULL SPACE  
↑  
of  $A$

$$A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$