EECS 245, Winter 2026

LEC 12 Rank, Column Space, and Null Space

→ Read Ch. 5.3 - 5.4

Agenda

- Recap: Matrices, transpose
- Four big ideas:
  - Column space
  - Rank
  - Null space
  - Rank-Nullity theorem

Announcements

- HW 6 due on Friday
- MT 1 regrades due tomorrow
- Sign up for a 1-on-1 check-in (email me if none of the slots work for you)
- Lab this week!

$$\|A\vec{x}\|^2 = (A\vec{x})\cdot(A\vec{x})$$
$$= (A\vec{x})^T(A\vec{x})$$
$$= \vec{x}^T A^T A\vec{x}$$

A : $n \times d$ matrix

cols

rows

$(AB)^T = B^T A^T$

$AB \neq BA$

$\vec{x}$: $\mathbb{R}^d$ ($d \times 1$ matrix)

vector

$A\vec{x}$: $\mathbb{R}^n$ ($n \times 1$ matrix)

vector

Aside: Transpose

$$A_{n\times d} \rightarrow A^T_{d\times n}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \vec{v}^T \vec{u} = \vec{v} \cdot \vec{u}$$

(not $\cancel{\vec{u}^T \cdot \vec{v}}$)

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$$\underbrace{A\vec{x} = (2)\begin{bmatrix} 5 \\ 0 \\ 3 \\ 6 \\ 1 \end{bmatrix} + (0)\begin{bmatrix} 3 \\ -1 \\ 4 \\ 2 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \\ 1 \end{bmatrix} \in \mathbb{R}^5}_{A\vec{x} \text{ is a lin. comb. of } A\text{'s columns!}}$$

column space of matrix A is

- the set of all linear combinations of A's columns
- i.e. span of A's columns
- set of outputs of $A\vec{x}$, for all possible $\vec{x}$'s

each column is in $\mathbb{R}^n$

$$A_{n\times d} = \begin{bmatrix} | & | & & | \\ \vec{a}^{(1)} & \vec{a}^{(2)} & \cdots & \vec{a}^{(d)} \\ | & | & & | \end{bmatrix}$$

$$\operatorname{colsp}(A) = \underbrace{\operatorname{span}\left(\left\{\vec{a}^{(1)}, \vec{a}^{(2)}, \ldots, \vec{a}^{(d)}\right\}\right)}_{\text{subspace of } \mathbb{R}^n}$$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\dim(\operatorname{colsp}(A)) = 2$$

important:

col 1 = col 2 + col 3

$$\operatorname{colsp}(A) = \operatorname{span}\left(\left\{ \underbrace{\begin{bmatrix} 5 \\ 0 \\ 3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \\ 1 \end{bmatrix}}_{\text{not linearly independent!}} \right\}\right)$$

$$= \operatorname{span}\left(\left\{ \underbrace{\begin{bmatrix} 3 \\ -1 \\ 4 \\ 2 \\ 0 \end{bmatrix}}_{\text{col 2}}, \underbrace{\begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \\ 1 \end{bmatrix}}_{\text{col 3}} \right\}\right) = \operatorname{span}(\{\text{col } 1, \text{col } 3\})$$

$$\operatorname{rank}(A) = \text{dimension of column space of } A$$

$$= \#\text{ linearly independent columns in } A$$

don't ever forget this!

1. 3×4 matrix of rank 1

rows cols

colsp is a line in $\mathbb{R}^3$

$$\begin{bmatrix} 1 & 2 & -1 & 10 \\ 1 & 2 & -1 & 10 \\ 2 & 4 & -2 & 20 \end{bmatrix}$$

" " rank 2

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

not all pairs of columns span the colsp, but colsp is spanned by just 2 cols

" " rank 3

$$\begin{bmatrix} 2 & 0 & 0 & 8 \\ 0 & 8 & 0 & 12 \\ 0 & 0 & 5 & 34 \end{bmatrix}$$

can't make 3×4 matrix with rank 4!

$$D = \begin{bmatrix} d_1 & & & & 0 \\ & d_2 & & & \\ & & d_3 & & \\ & 0 & & \ddots & \\ & & & & d_n \end{bmatrix}_{n \times n}$$

1) $d_1 = 1,\ d_2 = 2, \ldots,\ d_n = n$

$\mathrm{rank}(D) = n$

2) k of the d_i's $= 0$, rest non-zero

$\mathrm{rank}(D) = n - k$

$\vec{u}^T\vec{v} = 2-15-4 = -17 = \vec{u}\cdot\vec{v} \to$ dot product/inner product

cols multiples of $\vec{u}$, rows multiples of $\vec{v}^T$!

$$\vec{u} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}_{3\times 1} \qquad \vec{v} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}_{3\times 1}$$

"outer product" of $\vec{u}, \vec{v}$ ← vectors

$$\underset{3\times 1}{\vec{u}}\,\underset{1\times 3}{\vec{v}^T} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}\begin{bmatrix} 2 & 5 & -1 \end{bmatrix} = \begin{array}{c} \begin{array}{c} \\ \end{array} \\ \begin{bmatrix} 2 & 5 & -1 \\ -6 & -15 & 3 \\ 8 & 20 & -4 \end{bmatrix}_{3\times 3} \\ \begin{array}{ccc} 2\vec{u} & 5\vec{u} & -\vec{u} \end{array} \end{array} \begin{array}{l} \vec{v}^T \\ -3\vec{v}^T \\ 4\vec{v}^T \end{array}$$

$$\text{rank}(\vec{u}\vec{v}^T) = 1$$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 5 & 0 & 3 & 6 & 1 \\ 3 & -1 & 4 & 2 & 0 \\ 2 & 1 & -1 & 4 & 1 \end{bmatrix}$$

$\text{rowsp}(A)$ = span of A's rows!

$= \text{colsp}(A^T)$

row space is the set of all possible outputs of $A^T\vec{y}$, $\vec{y} \in \mathbb{R}^5$

"row space"

$$= \mathrm{colsp}(A^T)$$

$$
\begin{array}{c}
A^T = \begin{bmatrix} 5 & 0 & 3 & 6 & 1 \\ 3 & -1 & 4 & 2 & 0 \\ 2 & 1 & -1 & 4 & 1 \end{bmatrix} \\
\qquad\qquad \vec{a}_2 \qquad\qquad\qquad \vec{a}_5
\end{array}
$$

colsp(A^T) is a $\underline{2}$-dimensional subspace of $\mathbb{R}^3$

$\vec{a}_2, \vec{a}_5$ can span other cols of A^T

$$\text{rank}(A^T) = \cancel{3}\, 2 = \text{rank}(A)$$

Fact : For <u>any</u> matrix A,

\# of linearly independent columns

$=$

\# of linearly independent rows

i.e. $\text{rank}(A) = \text{rank}(A^T)$

A is an $n \times d$ matrix

$\text{rank}(A) = \text{rank}(A^T)$

rows can't be LI

$$
\begin{array}{c c}
n & \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \\ 0 & 4 \\ 0 & 6 \end{bmatrix} \\
 & d
\end{array}
$$

$n > d$

cols can't be LI

n

d

$n < d$

either **both** the cols & rows are LI, or neither

$n = d$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

since $A\vec{x} = \vec{0}$, $\vec{x}$ is in the NULL SPACE of A

$$A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$