EECS 245 Fall 2025
Math for ML

Lecture 12: Rank

→ Read 2.8 (new!)

Agenda

→ Overview: what's the point?

→ Column space and rank

→ Row space

→ Null space

→ (If time) CR decomposition

read Ch. 2.8!

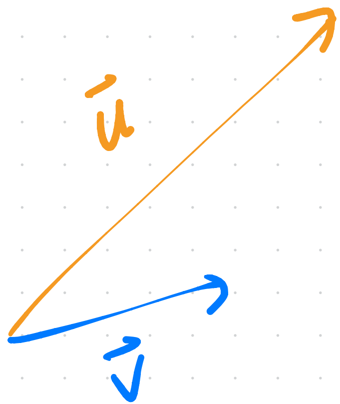
Announcements

: HW 5 due Thursday

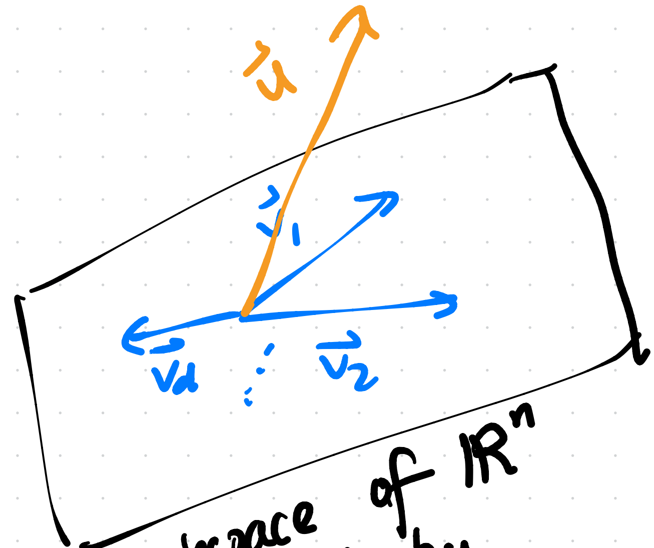
HW 6 out Friday, due next Friday

Midterm solutions posted; regrades due tomorrow

what's the point?



"approximation problem": just one $\vec{v}$



subspace of $\mathbb{R}^n$ spanned by $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d$

where we're heading soon!

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

5×3 $\quad A \in \mathbb{R}^{5\times 3}$

e.g. $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

$A\vec{x}$

must be $\mathbb{R}^3$

$(5 \times 3)(3 \times 1)$

$A\vec{x}$ is a vector in $\mathbb{R}^5$

AB

cols in A

=

rows in B

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$$\underbrace{A\vec{x} = 2\begin{bmatrix} 5 \\ 0 \\ 3 \\ 6 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -1 \\ 4 \\ 2 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \\ 1 \end{bmatrix} \in \mathbb{R}^5}_{A\vec{x} \text{ is a lin. comb. of A's columns}}$$

column space of A is the set of all linear combinations (span) of A's columns

$=$ set of possible outputs of $A\vec{x}$

$$A_{n\times d} = \begin{bmatrix} | & | & & | \\ \vec{a}^{(1)} & \vec{a}^{(2)} & \cdots & \vec{a}^{(d)} \\ | & | & & | \end{bmatrix}$$

$$\text{colsp}(A) = \underbrace{\text{span}\left(\vec{a}^{(1)}, \ldots, \vec{a}^{(d)}\right)}_{\text{subspace of } \mathbb{R}^n}$$

colsp(A) is a 2-dimensional subspace of $\mathbb{R}^5$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

col 3 = col 1 − col 2, it is a lin comb of other columns

$$\text{colsp}(A) = \text{span}\left(\left\{ \begin{bmatrix} 5 \\ 0 \\ 3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \\ 1 \end{bmatrix} \right\}\right)$$

$$= \text{span}\left(\left\{ \begin{bmatrix} 5 \\ 0 \\ 3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ 2 \\ 0 \end{bmatrix} \right\}\right)$$

$$\dim(\text{colsp}(A)) = 2$$

A has 2 linearly independent columns!

$$\text{rank}(A) = \text{dimension of the column space of } A$$

$$= \#\text{ of linearly independent columns in } A$$

don't ever forget this!

3 x 4 matrix with rank 1

rows columns

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

rank 2

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 5 \\ 1 & 1 & 1 & 9 \end{bmatrix}$$

rank 3

$$\begin{bmatrix} 1 & 0 & 0 & 17 \\ 0 & 1 & 0 & 24 \\ 0 & 0 & 1 & -98 \end{bmatrix}$$

why not rank 4?

can only have up to 3 lin. ind. vectors in $\mathbb{R}^3$

$$D = \begin{bmatrix} d_1 & & & & 0 \\ & d_2 & & & \\ & & d_3 & & \\ & 0 & & \ddots & \\ & & & & d_n \end{bmatrix}_{n\times n}$$

1) $d_1 = 1,\ d_2 = 2, \ldots, d_n = n$

$\text{rank}(D) = n$

2) k of the $d_i = 0$,

rest $= 1$?

$\text{rank}(D) = n - k$

"outer product" = "rank one matrix"

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$$

"dot product"

$$\vec{u} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$$

rows are multiples of $\vec{v}$

$$\vec{u}\vec{v}^T = \underset{2\times 1}{\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 2 & 5 & -1 \end{bmatrix}} = \begin{bmatrix} 2 & 5 & -1 \\ -6 & -15 & 3 \\ 8 & 20 & -4 \end{bmatrix}$$

columns are multiples of $\vec{u}$

$$\text{rank}(\vec{u}\vec{v}^T) = 1$$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

"row space"

$\text{rowsp}(A)$ = span of A's rows!

$$A^T = \begin{bmatrix} 5 & 0 & 3 & 6 & 1 \\ 3 & -1 & 4 & 2 & 0 \\ 2 & 1 & -1 & 4 & 1 \end{bmatrix}$$

$\vec{y} \in \mathbb{R}^5$

row space consists of all results of $A^T\vec{y}$, "row space"

$$\text{colsp}(A^T)$$

what is the dimension of A's row space?

i.e. what is $\text{rank}(A^T)$?

$$A^T = \begin{array}{c} \begin{array}{ccccc} & \vec{a_2} & & & \vec{a_5} \end{array} \\ \begin{bmatrix} 5 & 0 & 3 & 6 & 1 \\ 3 & -1 & 4 & 2 & 0 \\ 2 & 1 & -1 & 4 & 1 \end{bmatrix} \end{array}$$

$\text{colsp}(A^T)$ is 2-dim a subspace of $\mathbb{R}^3$, because any lin. comb. of A's rows is in $\mathbb{R}^3$.

$\dim(\text{colsp}(A^T)) = 2$, because $\vec{a_2}$ and $\vec{a_5}$ span all of it

Fact: for any matrix A,

\# of linearly independent columns

=

\# of linearly independent rows

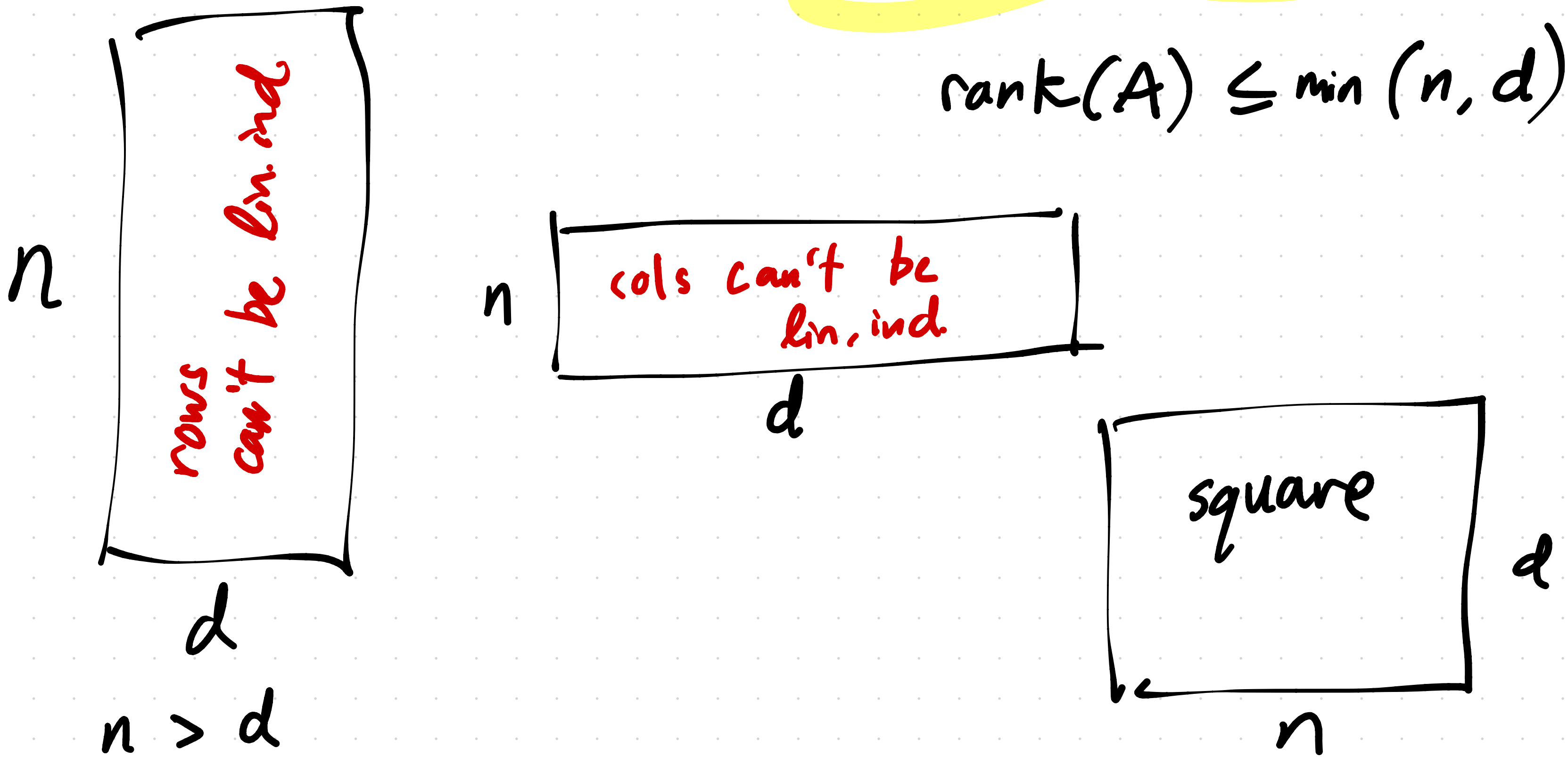
i.e.

$$\operatorname{rank}(A) = \operatorname{rank}(A^T)$$

A is $n \times d$ matrix

$\text{rank}(A) = \text{rank}(A^T)$

$\text{rank}(A) \leq \min(n, d)$



$n > d$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix} \qquad A\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \vec{0}$$

if A's columns are LI $\quad A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$'s

only solution is $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

otherwise other $\vec{x}$'s will get sent to $\vec{0}$

"null space" of $A^{n \times d}$ is the set of all $\vec{x}$'s such that

$$A\vec{x} = \vec{0}$$

$$\text{nullsp}(A) = \underbrace{\left\{ \vec{x} \in \mathbb{R}^{d} \;\middle|\; A\vec{x} = \vec{0} \right\}}_{\text{subspace of } \mathbb{R}^d}$$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A\vec{x} = \vec{0} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\mathrm{nullsp}(A)$

$$5x_1 + 3x_2 + 2x_3 = 0$$

$$-x_2 + x_3 = 0$$

$$\vdots$$

$$x_1 + x_3 = 0$$

$$x_2 = x_3$$

$$x_1 = -x_3$$

$$\text{nullsp}(A) = \left\{ \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} \; x_3 \in \mathbb{R} \right\}$$

$$= \text{span}\left(\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \right)$$

rank(A)

colsp(A): 2-d subsp of $\mathbb{R}^5$

colsp(A^T): 2-d subsp of $\mathbb{R}^3$

nullsp(A) = 1-d subsp of $\mathbb{R}^3$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

"rank-nullity" theorem: for any matrix $\underset{n \times d}{A}$,

$$\text{rank}(A) + \underbrace{\dim(\text{nullsp}(A))}_{\text{"nullity"}}$$

$$=$$

$$\# \text{ columns in } \underset{(d)}{A}$$

e.g. if $\text{rank}(A) = 1$,

nullsp(A) is plane in $\mathbb{R}^3$

(if A is 5×3)

1) column space,
row space,
null space

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

$\mathrm{rank}(A) = 1$

$\mathrm{colsp}(A)$ subspace of $\mathbb{R}^2$

$$= \mathrm{span}\left(\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}\right)$$

line in $\mathbb{R}^2$

row space

$\mathrm{colsp}(A^T)$ subspace of $\mathbb{R}^3$

$$= \mathrm{span}\left(\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}\right)$$

line in $\mathbb{R}^3$

$\mathrm{nullsp}(A)$ is a 2-d subspace of $\mathbb{R}^3$ (plane)

2) suppose A is a 7×9 matrix with rank 5.

rows columns

7 wide 9

rank + dim null = # cols

Find:

- $\dim(\overbrace{\text{colsp}(A)}^{\text{subsp of } \mathbb{R}^7}) = 5 = \text{rank}(A)$
- $\dim(\underbrace{\text{colsp}(A^T)}_{\text{subsp of } \mathbb{R}^9}) = 5$
- $\dim(\text{nullsp}(A)) = 9 - 5 = 4$
- $\dim(\underbrace{\text{nullsp}(A^T)}_{\text{subsp of } \mathbb{R}^7}) = 7 - 5 = 2$

cols in A^T = # rows in A