# EECS 245, Winter 2026

LEC 13 Null Space, Rank-Nullity, and Inverses

→ Read Ch. 5.4, 6.2

## Agenda

- Null space
- Rank-nullity theorem
- Inverse of a matrix

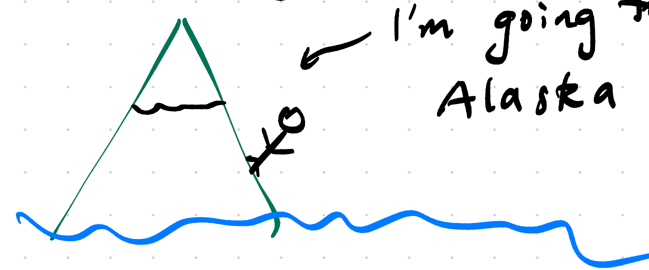
5.4 tons of examples, read!

6.2

We will skip Chapter 6.1 (linear transformations) for now and come back to it after the break.

## Announcements

- HW 6 due Friday
- There is lab tomorrow
- HW 7 out Friday, due Friday after break
- Next week is Spring Break, enjoy!



I'm going to Alaska

# Recap: rank and column space

Consider
$$A = \begin{bmatrix} 3 & 6 & 0 & 9 & 3 \\ 2 & 4 & 0 & 6 & 2 \\ 0 & 0 & 1 & -5 & 0 \\ 1 & 2 & 0 & 3 & 1 \end{bmatrix}$$

① rank(A) 2

same question!

② # linearly independent cols? rows? 2

③ 2 different bases for colsp(A)

$$A = \begin{bmatrix} 3 & 6 & 0 & 9 & 3 \\ 2 & 4 & 0 & 6 & 2 \\ 0 & 0 & 1 & -5 & 0 \\ 1 & 2 & 0 & 3 & 1 \end{bmatrix}$$

one basis for colsp(A) : $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

another : $\left\{ \begin{bmatrix} 6 \\ 4 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 90 \\ 60 \\ -50 \\ 30 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 3 & 6 & 0 & 9 & 3 \\ 2 & 4 & 0 & 6 & 2 \\ 0 & 0 & 1 & -5 & 0 \\ 1 & 2 & 0 & 3 & 1 \end{bmatrix}$$

$\vec{x} \in \mathbb{R}^5$

$$A\vec{x} = \vec{0}$$

linear combination of the columns of A!

**Important**: If A's columns are linearly independent, then the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

If other solutions exist, A's columns not linearly independent!

$$A = \begin{bmatrix} 3 & 6 & 0 & 9 & 3 \\ 2 & 4 & 0 & 6 & 2 \\ 0 & 0 & 1 & -5 & 0 \\ 1 & 2 & 0 & 3 & 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow A\vec{x} = \vec{0}$$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -2 \end{bmatrix} \Rightarrow A\vec{x} = \vec{0}$$

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ -1 \\ 3 \end{bmatrix} \Rightarrow A\vec{x} = \vec{0}$$

## null space of $A$

Suppose $A$ is an $n \times d$ matrix.

$$\mathrm{nullsp}(A) = \underbrace{\left\{ \vec{x} \in \mathbb{R}^d \;\middle|\; A\vec{x} = \vec{0} \right\}}_{\text{set if all } \vec{x}\text{'s such that } A\vec{x} = \vec{0}}$$

$\Rightarrow$ subspace of $\mathbb{R}^d$

$$A = \begin{bmatrix} 3 & 6 & 0 & 9 & 3 \\ 2 & 4 & 0 & 6 & 2 \\ 0 & 0 & 1 & -5 & 0 \\ 1 & 2 & 0 & 3 & 1 \end{bmatrix}$$

$\text{rank}(A) = 2 = \dim(\text{colsp}(A))$

What is the dimension of nullsp$(A)$?

$\Rightarrow$ must be $\leq 5$, since nullsp$(A)$ is a subspace of $\mathbb{R}^5$

## Rank-nullity theorem

Suppose $A$ is an $n \times d$ matrix. Then,

$$\mathrm{rank}(A) + \dim(\mathrm{nullsp}(A)) = \underbrace{d}_{\#\text{ columns in } A}$$

$$A = \begin{bmatrix} 3 & 6 & 0 & 9 & 3 \\ 2 & 4 & 0 & 6 & 2 \\ 0 & 0 & 1 & -5 & 0 \\ 1 & 2 & 0 & 3 & 1 \end{bmatrix}$$

$\mathrm{rank}(A) = 2$

$d = 5$

so, $\dim(\mathrm{nullsp}(A)) = 3$

## Example

$$B = \begin{array}{ccc} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \\ \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \end{array}$$

$\text{rank}(B) = 1$

$\vec{v}_2 = 2\vec{v}_1$

$\dim(\text{nullsp}(B)) = 2$

$2\vec{v}_1 - \vec{v}_2 + 0\vec{v}_3 = \vec{0}$

basis for $\text{nullsp}(B)$

$$= \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix} \right\}$$

just one of the $\infty$ possible bases, but all bases have only 2 vectors!

$\Rightarrow B \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \vec{0}$,

so $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \in \text{nullsp}(B)$

$$\text{basis for nullsp}(B) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix} \right\}$$

any LC of $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix}$ is in nullsp$(B)$!

e.g. $15 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ -15 \\ 6 \end{bmatrix} \Rightarrow B \begin{bmatrix} 12 \\ -15 \\ 6 \end{bmatrix} = \vec{0}$

$\begin{bmatrix} 12 \\ -15 \\ 6 \end{bmatrix} \in \text{nullsp}(B)$

## Example

Find a matrix $A$ such that

$$\begin{bmatrix}3\\1\end{bmatrix} \in \operatorname{colsp}(A) \quad \text{and}$$

$$\begin{bmatrix}3 & 6 & 0 & 12 & 3\\1 & 2 & 0 & 15 & 1\end{bmatrix}$$

$$\begin{bmatrix}1\\3\end{bmatrix} \in \operatorname{nullsp}(A)$$

$$\begin{bmatrix}-12 & 4\\-12 & 4\\-12 & 4\\-12 & 4\end{bmatrix}$$

tells you A has 2 rows

tells you A has 2 cols

$[3, 1]^T$ must be a LC of A's columns; doesn't need to be a column itself

A's cols aren't linearly independent!

$$A = \begin{bmatrix}-9 & 3\\-3 & 1\end{bmatrix}_{2\times 2} \qquad A = \begin{bmatrix}6 & -2\\2 & -2/3\end{bmatrix}$$

**Example**: $\overset{n\ \times\ d}{7\times 9}$ matrix, $A$, $\text{rank}(A) = 5$

Dimensions of ....

$\text{colsp}(A)$: 5-d subspace of $\mathbb{R}^7$

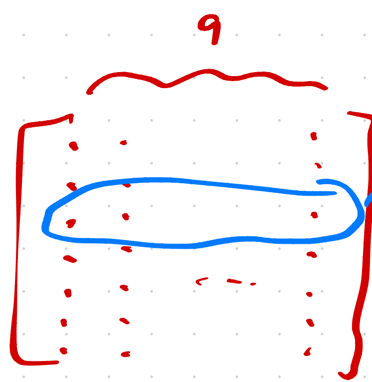
$\text{nullsp}(A)$: 4-d subspace of $\mathbb{R}^9$

$\underbrace{\text{colsp}(A^T)}_{\text{row space of } A}$: 5-d subspace of $\mathbb{R}^9$

$\text{nullsp}(A^T)$: 2-d subspace of $\mathbb{R}^7$

$\{\vec{y} : A^T\vec{y} = \vec{0}\}$

$\vec{y}^T A = \vec{0}$




each row is in $\mathbb{R}^9$

$$\text{rank}(A) + \dim(\text{nullsp}(A)) = \#\text{ cols of } A$$

$$\Downarrow$$

$$\underset{\text{rank}(A)}{\underset{\parallel}{\text{rank}(A^T)}} + \dim(\text{nullsp}(A^T)) = \#\text{ cols of } A^T$$

$$= \underbrace{\#\text{ rows of } A}_{n}$$

$$\Rightarrow \quad \dim(\text{nullsp}(A^T)) = n - \text{rank}(A)$$

2 videos, stay tuned

1. orthogonal complements

2. $\operatorname{rank}(X^T X) = \operatorname{rank}(X)$

## Briefly, inverses

scalar addition:

$$5 + (-5) = 0$$

adding 0 doesn't change a value!

scalar multiplication:

$$(7)(7)^{-1} = 1$$

1/7

$0^{-1}$ doesn't exist!

# Matrix inverses (multiplication)

$A^{-1}$ is the inverse of $A$.

→ It should satisfy

$$AA^{-1} = A^{-1}A = I$$

multiplying by identity doesn't change value

→ $A$ must be ___square___ to have an inverse!

Only square matrices CAN be invertible

---- but not all are!

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

NOT invertible!

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

invertible, because cols are linearly independent!

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \vec{b}$$

→ can $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be made? no!

→ can $\vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ be made?

yes, but $\infty$ many ways!

$A$ is invertible when there is EXACTLY one $\vec{x}$ such that $A\vec{x} = \vec{b}$, for all possible $\vec{b}$'s