EECS 245 Fall 2025
Math for ML

Lecture 13: Inverses

→ Read Ch 2.9 (new!)

Agenda

Review of 2.8

→ Recap: rank, col space, null space, rank-nullity

→ What is an inverse?

→ Linear transformations (and inverting them)

→ Inverse of a matrix

Ch. 2.9

Prove that $\operatorname{rank}(X^TX) = \operatorname{rank}(X)$

where X is an $n \times d$ matrix

⓪ what is rank?

① think about their shapes

② rank-nullity theorem

⓪ rank = # linearly independent cols

① shapes: X $\quad n \times d$

$\underset{d\times n}{X^T}\underset{n\times d}{X}$ $\quad d \times d$

same # of cols

② $\operatorname{rank}(A) + \dim(\operatorname{nullsp}(A)) = $ # cols of A

to X: $\operatorname{rank}(X) + \dim(\operatorname{nullsp}(X)) = d$

to X^TX: $\operatorname{rank}(X^TX) + \dim(\operatorname{nullsp}(X^TX)) = d$

Goal: show $\dim(\text{nullsp}(X)) = \dim(\text{nullsp}(X^T X))$

Even better: we can show $\underbrace{\text{nullsp}(X)}_{\text{a set "A"}} = \underbrace{\text{nullsp}(X^T X)}_{\text{a set "B"}}$

Strategy:

① Show that if $\vec{v} \in \text{nullsp}(X)$,
then $\vec{v} \in \text{nullsp}(X^T X)$

② show that if $\vec{v} \in \text{nullsp}(X^T X)$,
then $\vec{v} \in \text{nullsp}(X)$

need to show both!

① $\vec{v} \in \text{nullsp}(X) \longrightarrow \vec{v} \in \text{nullsp}(X^T X)$

means that

$$X\vec{v} = \vec{0}$$

multiply both sides by X^T!

$$X^T X \vec{v} = X^T \vec{0}$$

$$X^T X \vec{v} = \vec{0}$$

$\Rightarrow \vec{v}$ is in $\text{nullsp}(X^T X)$ too!

② $\underbrace{\vec{v} \in \text{nullsp}(X^T X)}_{\text{start with}} \longrightarrow \vec{v} \in \text{nullsp}(X)$

$X : n \times d$

$X^T : d \times n$

$(AB)^T = B^T A^T$

$$\underset{d \times n}{X^T} X \underset{d \times 1}{\vec{v}} = \vec{0}$$

hmm.... what if we multiply both sides by $\vec{v}^T$!

$$\vec{v}^T X^T X \vec{v} = \vec{v}^T \vec{0} = \underbrace{0}_{\text{scalar}}$$

$$(X\vec{v})^T X \vec{v} = 0$$

$$(X\vec{v}) \cdot (X\vec{v}) = 0$$

$$\|X\vec{v}\|^2 = 0 \Longrightarrow X\vec{v} \text{ must be } \vec{0}!$$

Inverse

$3 + a = 0$

$a = -3$

= "additive inverse" of 3

$\left(\frac{1}{4}\right)4x = 5\left(\frac{1}{4}\right)$

$x = \frac{5}{4}$

use inverse to solve systems of equations

$(-2)a = 1$

$a = \frac{1}{-2}$

$\downarrow$

$0a = 1$

has no solution!

no value for $\frac{1}{0}$, 0 has no inverse

A $n \times d$ matrix

inverse A'

ideally, $\underset{n\times d}{A}\,\underset{d\times\_}{A'} = \underset{\_\times n}{A'}\,\underset{n\times d}{A} = \underbrace{I}_{\text{identity matrix}}$

$A' = d \times n$
doesn't work bc

$AA' : n \times n$ $\quad$ $A'A : d \times d$

need to be same!

→ key takeaway: only square matrices are invertible!

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$IA = AI = A$$

e.g.

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \qquad A^{-1} = \underbrace{\begin{bmatrix} -5/2 & 2 \\ 3/2 & -1 \end{bmatrix}}_{\text{unique matrix such that}}$$

$$A\underbrace{A^{-1}}_{\text{inverse of } A} = A^{-1}A = I$$

this A is invertible

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{no } B^{-1} \text{ exists!}$$

$\text{rank}(B) = 1 < \#\text{cols}\ (2)$

Linear transformation

$f(x) = 2x+3$ ~~not~~ **not** a lin. trans!

$c=2, x=3$ $\quad f(6) = 15 \neq 2f(3) = 2[9] = 18$

function T is a linear transformation if

① $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ for any valid $\vec{x}, \vec{y}$

② $T(c\vec{x}) = c\,T(\vec{x})$ for any valid $\vec{x}$, and $c \in \mathbb{R}$

for us,

linear transformations

=

matrix-vector multiplications

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ -x_1 \\ 4x_1 - 2x_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 4 & -2 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

big idea:

multiplying A by $\vec{x}$ "transforms" $\vec{x}$ into a new vector.

A square: doesn't change dimension of vector

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbb{R}^2} = \underbrace{\begin{bmatrix} \\ \end{bmatrix}}_{\mathbb{R}^2}$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1/3 \end{bmatrix} \qquad T(\vec{x}) = A\vec{x}$$

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

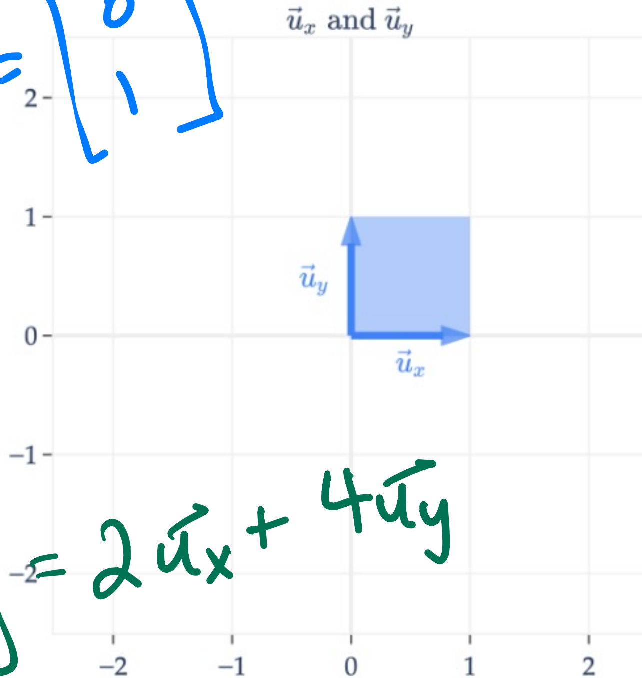
what does A do to $\vec{x}$?

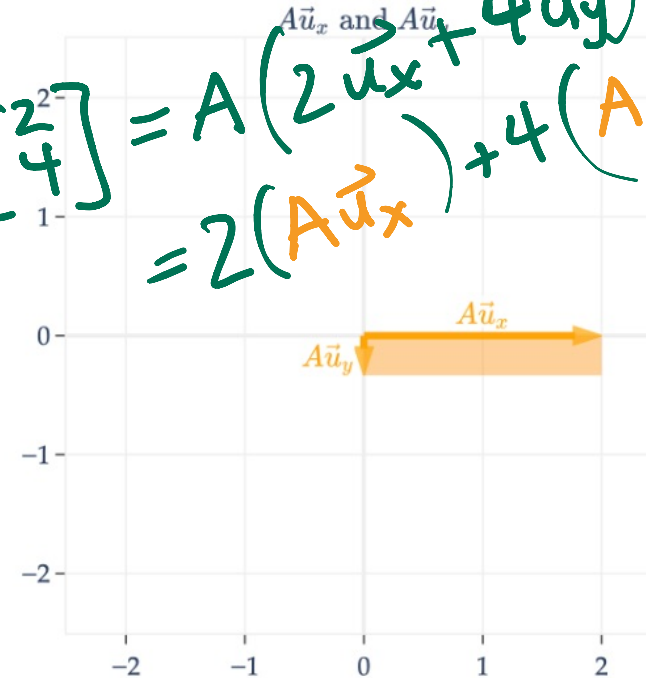
"scaling" "stretching"

$$\vec{u}_x = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\vec{u}_y = \begin{bmatrix}0\\1\end{bmatrix}$$

$$A = \begin{bmatrix}2 & 0\\0 & -1/3\end{bmatrix}$$



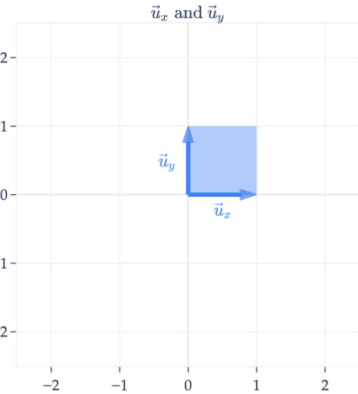


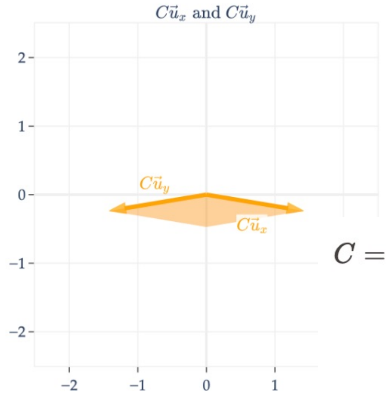
$$A\begin{bmatrix}2\\4\end{bmatrix} = A(2\vec{u}_x + 4\vec{u}_y)$$

$$= 2(A\vec{u}_x) + 4(A\vec{u}_y)$$

$$\begin{bmatrix}2\\4\end{bmatrix} = 2\vec{u}_x + 4\vec{u}_y$$

$$C = AB = \underbrace{\begin{bmatrix}2 & 0\\ 0 & -1/3\end{bmatrix}}_{\text{scale}}\underbrace{\begin{bmatrix}\sqrt{2}/2 & -\sqrt{2}/2\\ \sqrt{2}/2 & \sqrt{2}/2\end{bmatrix}}_{\text{rotate}}$$

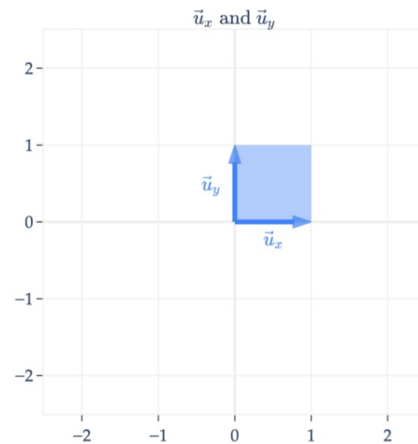


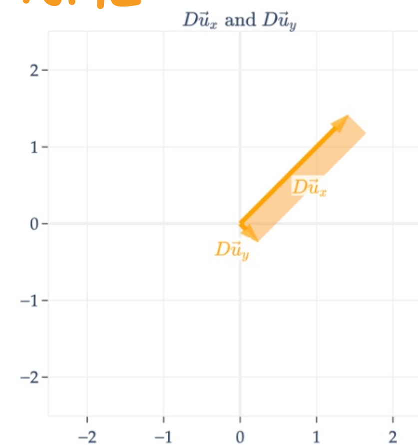


$C = AB$ is different from

$$D = \begin{bmatrix}\sqrt{2} & \sqrt{2}/6\\ \sqrt{2} & -\sqrt{2}/6\end{bmatrix} = BA$$

rotate scale



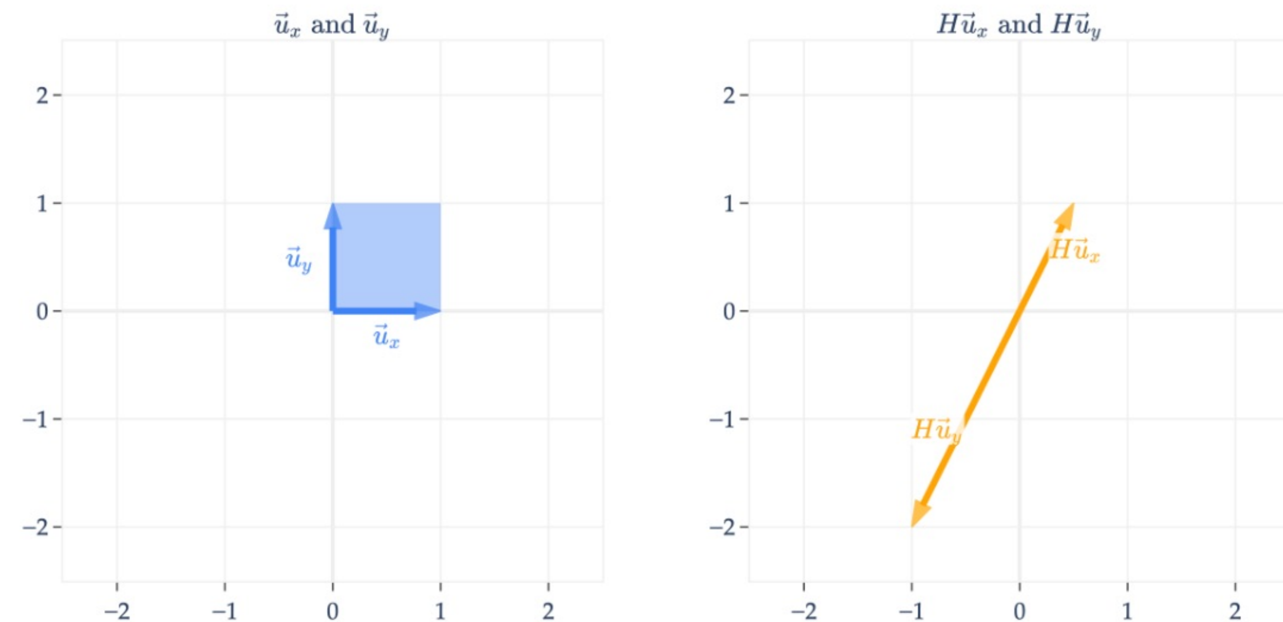


$$B = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

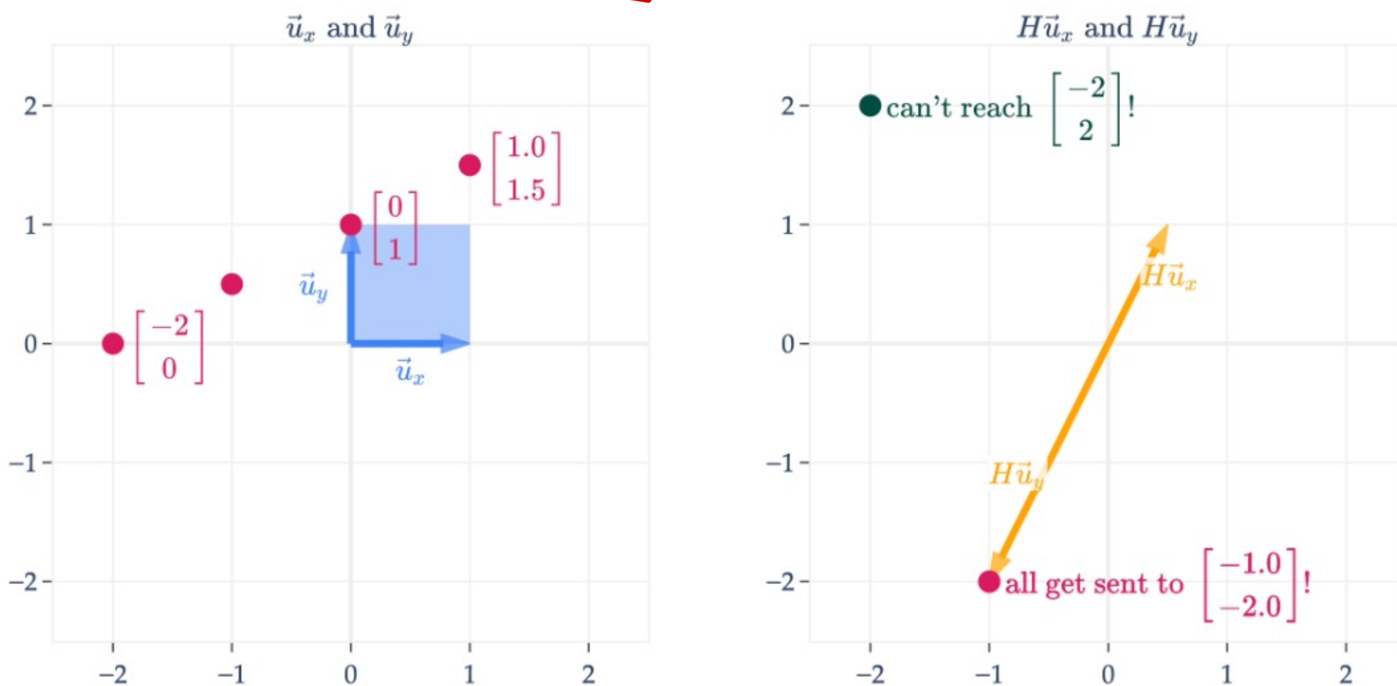
H below works similarly.

$$H = \begin{bmatrix} 1/2 & -1 \\ 1 & -2 \end{bmatrix}$$

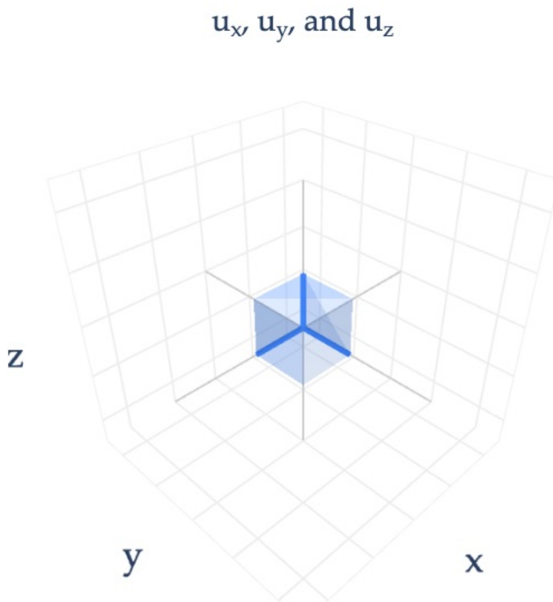
$\text{colsp}(H) = \text{span}\left(\left\{\begin{bmatrix} -1 \\ -2 \end{bmatrix}\right\}\right)$

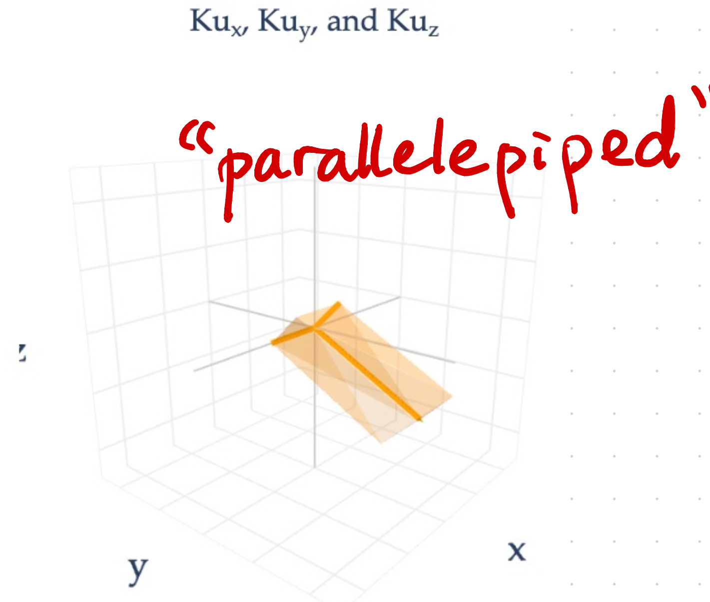


$$H = \begin{bmatrix} 1/2 & -1 \\ 1 & -2 \end{bmatrix}$$



$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1/2 \\ 0 & -1 & 1/2 \end{bmatrix}$$





big idea:

A's columns are linearly independent $\Longleftrightarrow$ A has an inverse

square

non-example

infinitely many $\vec{x}$!

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 5 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$\text{rank}(A) = 2$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 5 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 4 \\ 12 \end{bmatrix}$$

no such $\vec{x}$ exists!

$A_{n\times n}$ is invertible if ANY of the following equivalent conditions hold:

① A's cols are linearly independent

② $\text{rank}(A) = n$ (number of cols) "full rank"

③ $\text{colsp}(A) = \mathbb{R}^n$

④ $\text{colsp}(A^T) = \mathbb{R}^n$

⑤ $\text{nullsp}(A) = \{\vec{0}\}$

$$\underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & -9 \\ 3 & 4 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{A^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$2a + 3d + g = 1$$
$$a + \_\_ - 9g = 0$$
$$3a + 4d + 2g = 0$$

$$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

which input $\vec{x}$ gets me to $\vec{b}$?

$$\vec{x} = A^{-1}\vec{b}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

2x2 case is the only one worth memorizing

$$AA^{-1} = I$$

$$A^{-1}A = I$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$