EECS 245, Winter 2026

LEC 14 Inverses and Linear Transformations

→ Read: Ch. 6.1 - 6.2

Agenda

Read 6.1–6.2!

Goal: deeply understand the conditions in which a matrix is invertible

→ Recap: what is an inverse?

→ Linear transformations, determinants

→ More inverse exercises

Announcements

- HW 7 due Friday
- Lots of supplemental videos posted on website:
  - rank-nullity theorem (linked next to last lec)
  - determinants (linked next to today's lec)
  - WN26 MT1 walkthroughs (resources tab)


Alaska
AIRLINES
Cordova

11:58 AM Mon Mar 9

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Suraj Rampure

My Flight Log

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San Francisco to Detroit

AS 680 SEA → SFO — Mar 4, 2026
Seattle to **San Francisco**

AS 60 KTN → SEA — Mar 4, 2026
Ketchikan to **Seattle**

AS 64 WRG → KTN — Mar 3, 2026
Wrangell to **Ketchikan**

AS 64 PSG → WRG — Mar 3, 2026
Petersburg to **Wrangell**

AS 64 JNU → PSG — Mar 3, 2026
Juneau to **Petersburg**

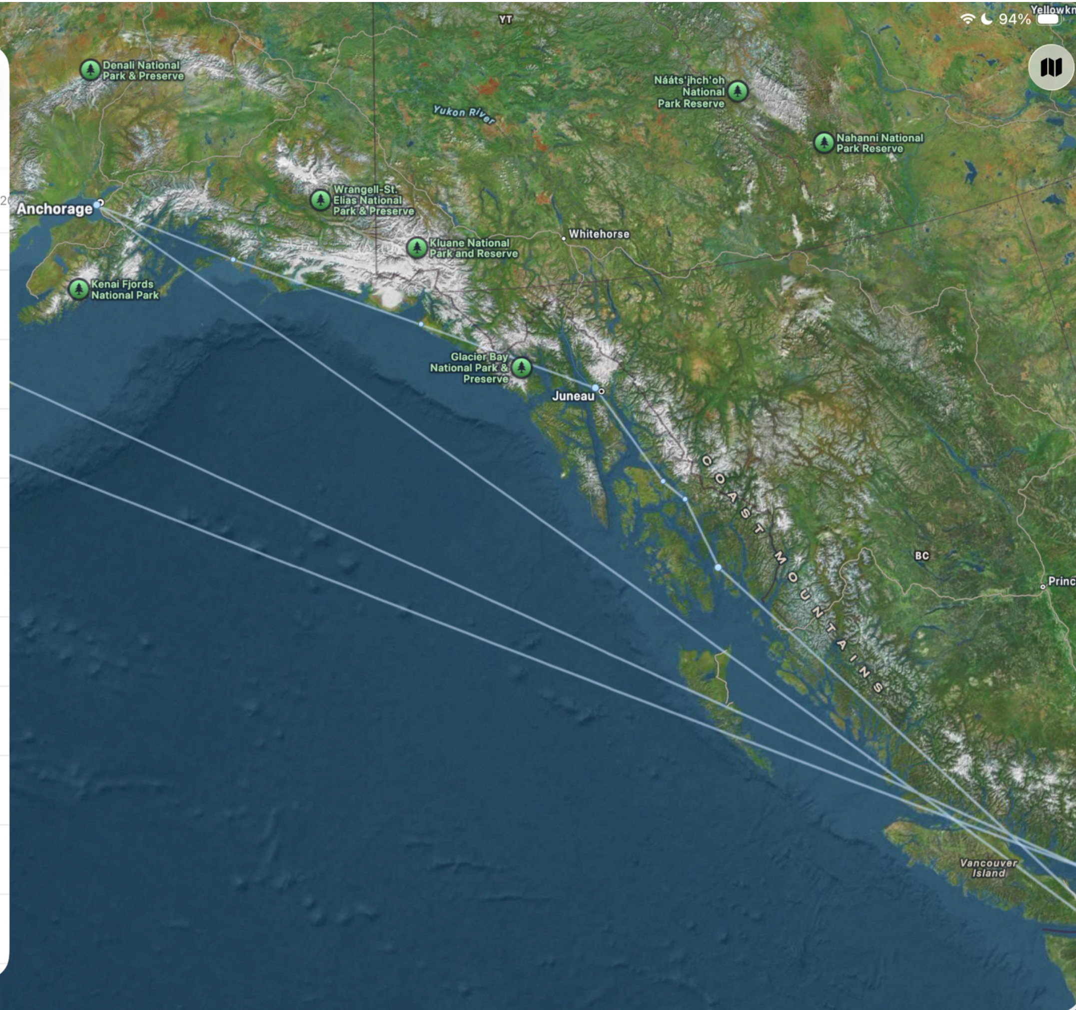
AS 66 YAK → JNU — Mar 2, 2026
Yakutat to **Juneau**

AS 66 CDV → YAK — Mar 2, 2026
Cordova to **Yakutat**

AS 66 ANC → CDV — Mar 2, 2026
Anchorage to **Cordova**

DL 922 SEA → ANC — Feb 28, 2026
Seattle to **Anchorage**

DL 728 DTW → SEA — Feb 27, 2026
Detroit to **Seattle**



Maps Legal

Suppose A is an invertible matrix.

What does that mean.....?

→ A must be square!

→ there exists a <u>unique</u> matrix A^{-1}

such that

$$AA^{-1} = A^{-1}A = I$$

Example : 2×2 matrix

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 12 \end{bmatrix}$$

if this was 10, A not invertible!

$$\Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} 12 & -5 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5/2 \\ -1 & 1/2 \end{bmatrix}$$

inverse of 2×2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant

Example : 3x3 matrix

$$\underbrace{\begin{bmatrix} 5 & 2 & 3 \\ 7 & 1 & 4 \\ 2 & 0 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

$$5a + 2d + 3g = 1$$

$$\vdots$$

could find A^{-1} by solving systems!

A is an $n \times n$ matrix.

A is invertible if <u>any</u> of these equivalent conditions hold:

- $\text{rank}(A) = n$
- A's columns are linearly independent
- A's rows are linearly independent
- $\det(A) \neq 0$

Linear transformation

Suppose T is a function.

T is a linear transformation if it satisfies these 2 properties:

- $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$
- $T(c\vec{x}) = cT(\vec{x})$

e.g.

$f(x) = 2x+5$

NOT A LIN TRANS!

e.g. $f(6)$ vs $3f(2)$

$= 17$ $= 27$

$$T\left(\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}\right) = \begin{bmatrix} x_1 + x_2 \\ 2x_1 \\ 3x_1 - 4x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}_{3\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\vec{x}$$

<u>big idea</u>: all linear transformations are matrix-vector multiplications!

$$T(\vec{x}) = \underset{n\times d}{A}\,\vec{x}_{d\times 1}$$

$$T : \underbrace{\mathbb{R}^d}_{\text{domain}} \to \underbrace{\mathbb{R}^n}_{\text{codomain}}$$

if A is square, $d=n$, so

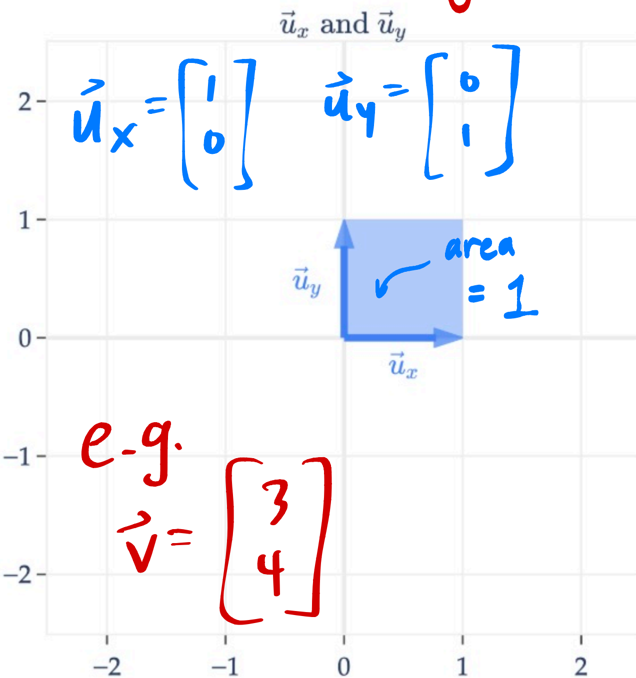
$$T: \mathbb{R}^n \to \mathbb{R}^n$$

$\Rightarrow$ focus on square matrices!

A "stretches" each component individually!

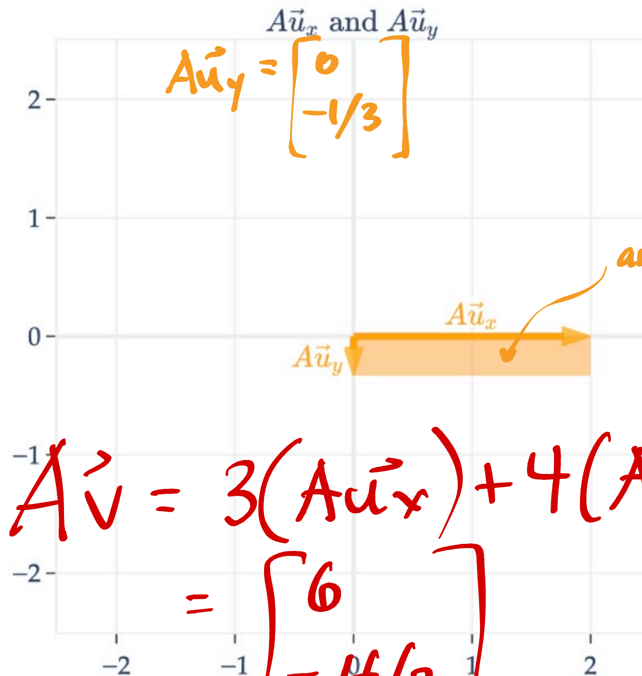
$$A = \begin{bmatrix}2 & 0\\ 0 & -1/3\end{bmatrix}$$

$$A\vec{u}_x = A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$$



$\vec{u}_x = \begin{bmatrix}1\\0\end{bmatrix}$ $\vec{u}_y = \begin{bmatrix}0\\1\end{bmatrix}$

area $= 1$



$A\vec{u}_y = \begin{bmatrix}0\\-1/3\end{bmatrix}$

area $= 2/3$

e.g.

$\vec{v} = \begin{bmatrix}3\\4\end{bmatrix}$

$$A\vec{v} = 3(A\vec{u}_x) + 4(A\vec{u}_y) = \begin{bmatrix}6\\-4/3\end{bmatrix}$$

Suppose $\vec{v} = c\vec{u}_x + d\vec{u}_y = \begin{bmatrix} c \\ d \end{bmatrix}$

Then,

$$A\vec{v} = A(c\vec{u}_x + d\vec{u}_y)$$
$$= \underbrace{c}(A\vec{u}_x) + \underbrace{d}(A\vec{u}_y)$$

$A\vec{v}$ is a linear combination of $A\vec{u}_x$ and $A\vec{u}_y$!

B "rotates"

$\|B\vec{v}\| = \|\vec{v}\|$

$$B = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

orthogonal matrix!

$B^T B = B B^T = I$

C rotates, then scales!

$$C = AB = \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & -1/3 \end{bmatrix}}_{\text{scale}} \underbrace{\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}}_{\text{rotate}} \vec{v}$$

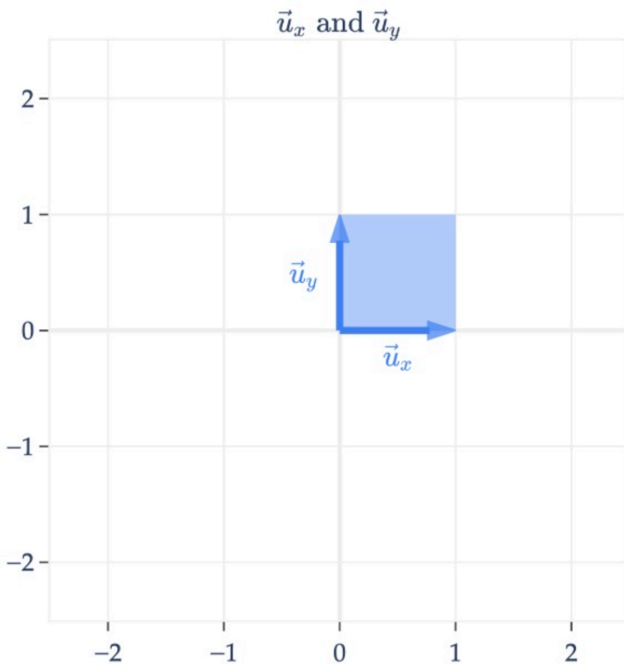
$C\vec{v}$

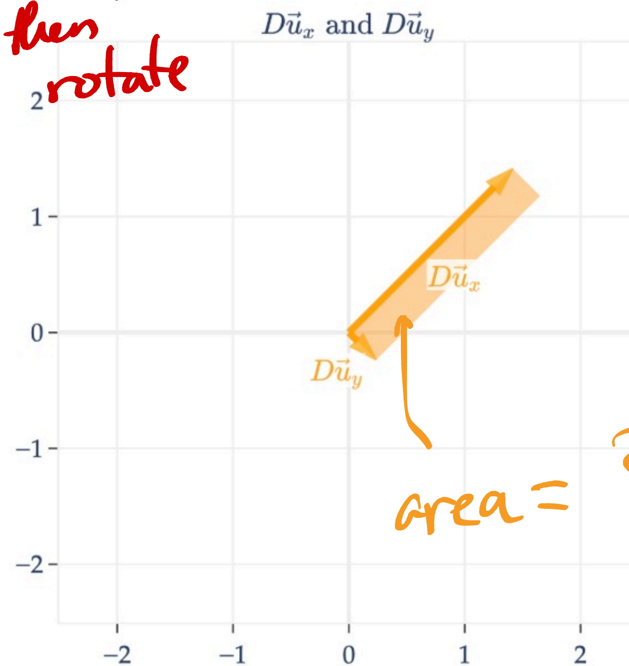
$C = AB$ is different from

$$D = \begin{bmatrix} \sqrt{2} & \sqrt{2}/6 \\ \sqrt{2} & -\sqrt{2}/6 \end{bmatrix} = BA$$

then rotate

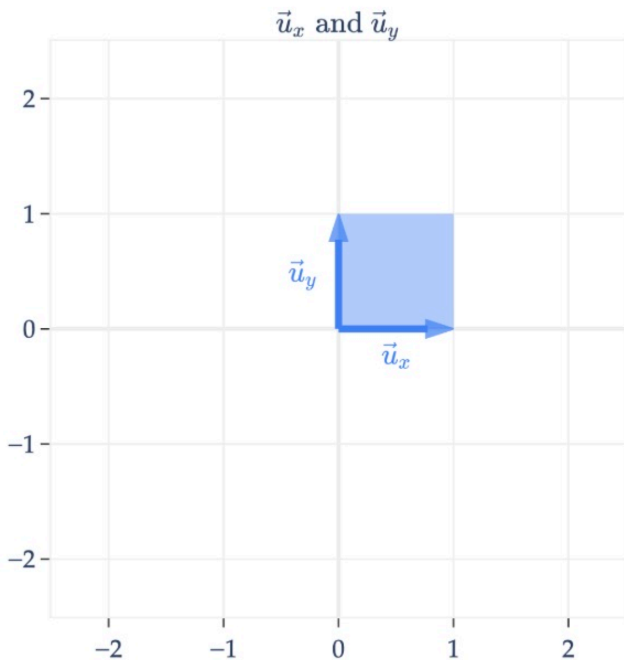
scale

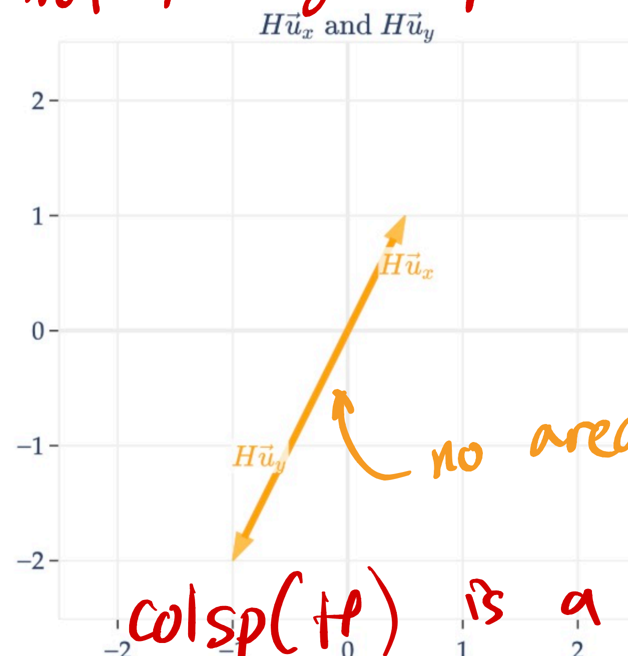




area = ?

$$H = \underbrace{\begin{bmatrix} 1/2 & -1 \\ 1 & -2 \end{bmatrix}}_{\text{not linearly independent!}}$$

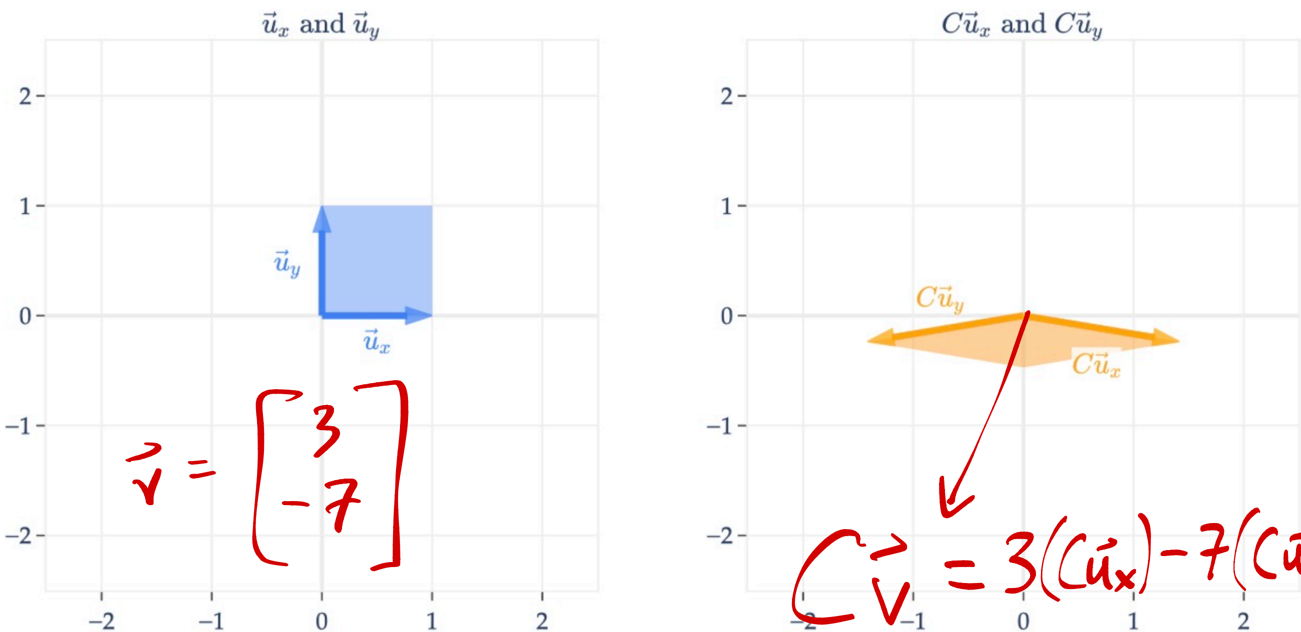




no area!

colsp(H) is a line!

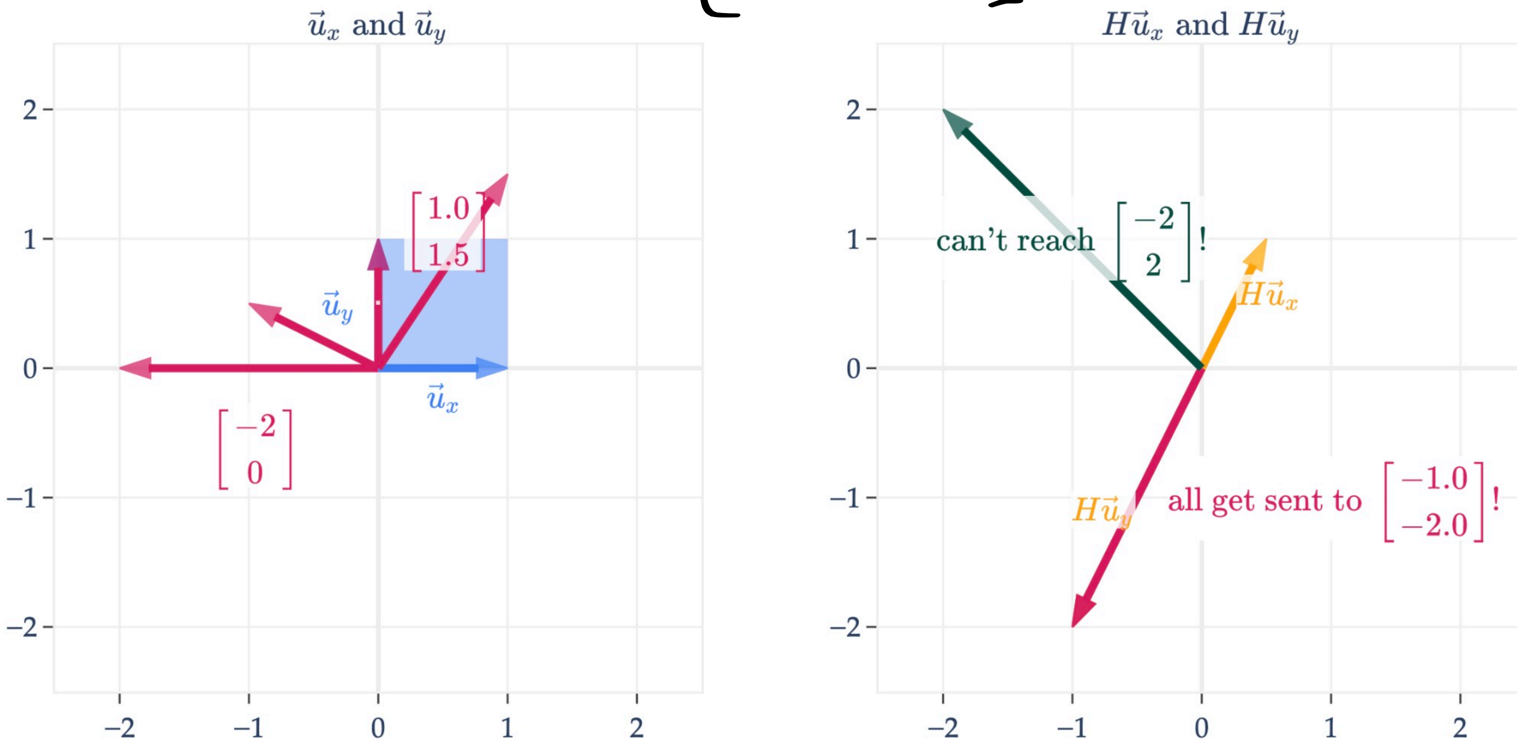
$$C = AB = \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & -1/3 \end{bmatrix}}_{\text{scale}} \underbrace{\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}}_{\text{rotate}}$$



$$\vec{v} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

$$C\vec{v} = 3(C\vec{u}_x) - 7(C\vec{u}_y)$$

$$H = \begin{bmatrix} 1/2 & -1 \\ 1 & -2 \end{bmatrix}$$



big idea: $n \times n$ matrix A is invertible

if and only if

the transformation $T(\vec{x}) = A\vec{x}$

is invertible

$$A\vec{x} = \vec{b}$$ ← linear combination of A's cols

system of n equations, n unknowns $(x_1, x_2, \ldots, x_n)$

Q: what linear combination — that is, which coefficients $x_1, x_2, \ldots, x_n$ — make $\vec{b}$?

⇒ big idea: if A invertible, there is exactly one $\vec{x}$ for every $\vec{b}$

$$A^{-1} A \vec{x} = A^{-1} \vec{b}$$

$$\vec{x} = A^{-1} \vec{b}$$

practical consideration: finding A^{-1} can be challenging, so we prefer to solve systems directly

read 6.2

Determinant

$$J = \begin{bmatrix} 1/3 & -1 \\ 1 & -1/2 \end{bmatrix}$$