EECS 245 Fall 2025
Math for ML

Lecture 14: Inverses, Projections

→ Read : Ch 2.9 (new examples added!)
Ch 2.10 (in progress)

Agenda

1. Recap: Inverses (review of last lecture and lab)
2. Projecting onto the span of multiple vectors

final idea before we go back to "core" machine learning!

Announcements:

1. HW 6 due Monday → do it, don't drop it!
2. Check the "Grade Report" on Gradescope

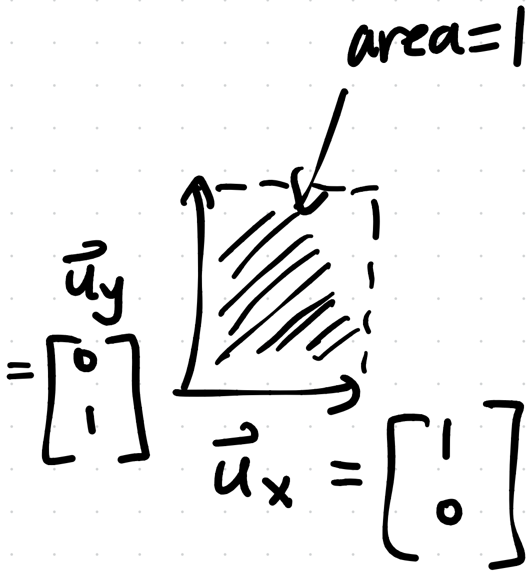
Inverses

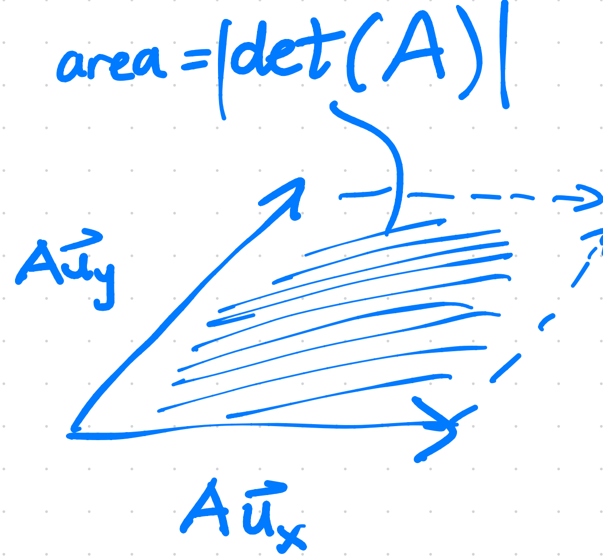
square

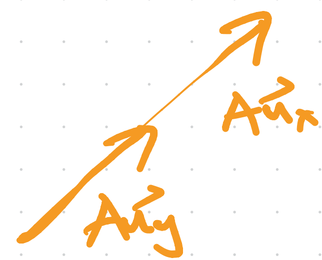
Suppose A is an $n \times n$ matrix that is invertible

$\Rightarrow$ this means there exists another $n \times n$ matrix B such that

$$\underbrace{AB}_{B \text{ is } A\text{'s ``right inverse''}} = BA = I = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}$$







$\det(A) = 0 \Rightarrow$ A's columns are dependent $\rightarrow$ A <u>not</u> invertible

If A^2 is invertible, prove A is invertible

Three approaches : 1. Constructing A's inverse

2. Null spaces

3. Determinant

① Constructing A's inverse

given A^2 invertible : what does that mean?
means there exists matrix B where

$$A^2 B = BA^2 = I$$

$AAB = I$ ①
$BAA = I$ ②

can we use this to find A's inverse?

in ①, $A(AB) = I \rightarrow A^{-1} = AB$
in ②, $(BA)A = I \rightarrow A^{-1} = BA$

→ which is it? AB or BA?
→ in general, they are different, but here, $AB = BA$

proof that here, $AB = BA$

$I^{-1} = I$

start with BA

$$BA = BAI = BA\underbrace{AAB}_{=I \text{, from last slide}} = \underbrace{(BAA)}_{\text{also } I!}AB = IAB$$

$$= AB$$

so, $A^{-1} = AB = BA$

② Null spaces

if A^2 invertible, then A is invertible

start by assuming the statement isn't true,

assume A^2 invertible but A isn't

→ if A isn't invertible, then there exists some $\vec{x} \neq \vec{0}$ where

$$A\vec{x} = \vec{0}$$

→ but then $AA\vec{x} = A\vec{0} = \vec{0}$ → $A^2\vec{x} = \vec{0}$

→ contradiction since we assumed A^2 inv.

recall,

rank + dim nullsp = # cols

③ Determinants

A^2 invertible $\longrightarrow$ A invertible

if A^2 invertible, $\det(A^2) \neq 0$

$$\det(A^2) \neq 0$$

$$\det(AA) \neq 0$$

$$\det(A)\det(A) \neq 0$$

$$[\det(A)]^2 \neq 0$$

$$\det(A) \neq 0 \longrightarrow A \text{ is invertible}$$

$$\det(AB) = \det(A)\det(B)$$

$$\det(A+B) \neq \det(A)+\det(B)$$

if $A\begin{bmatrix}3\\1\\-1\end{bmatrix}=\begin{bmatrix}9\\4\\3\end{bmatrix}$ and $\underbrace{A\begin{bmatrix}4\\0\\2\end{bmatrix}}_{\text{lin comb of A's cols}}=\begin{bmatrix}0\\0\\0\end{bmatrix}$,

is A invertible?

no! $\begin{bmatrix}4\\0\\2\end{bmatrix}\in \text{nullsp}(A)$, which

means $\dim(\text{nullsp}(A)) \geq 1$,

means $\text{rank}(A) \leq 2$

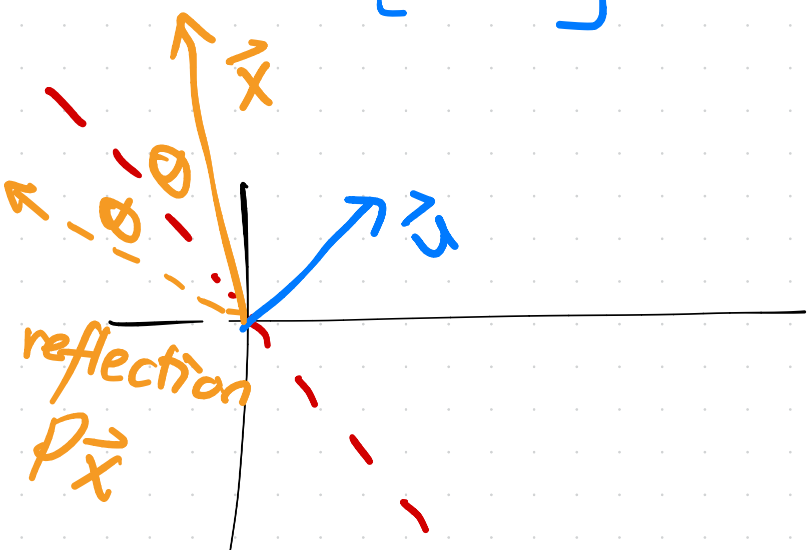
Unit vector $\vec{u}$

consider the equation $\vec{u}\cdot\vec{x} = 0$ = line $y = -\frac{3}{4}x$

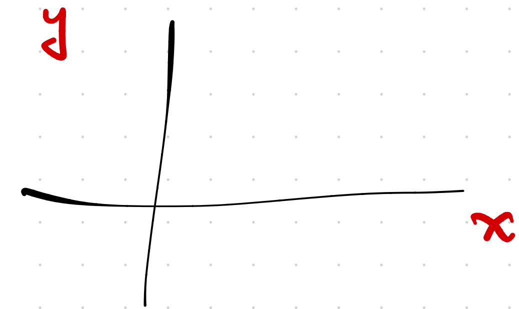
e.g. $\vec{u} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

$$\frac{3}{5}x + \frac{4}{5}y = 0$$

$$y = -\frac{3}{4}x$$



Line $\vec{u}\cdot\vec{x} = 0$



$$P = \underbrace{I}_{n \times n \text{ matrix}} - 2\underbrace{\vec{u}\vec{u}^T}_{n \times n \text{ matrix}}$$

Householder "reflection matrix"

$P\vec{x}$ is the reflection of $\vec{x}$ across the line/plane/

$$\vec{u} \cdot \vec{x} = 0$$

P is orthogonal! $\underbrace{PP^T = P^TP = I}_{\text{equivalent to}}$

$$P^{-1} = P^T$$

also, $P^T = P$ $\Rightarrow P^{-1} = P^T = P \Rightarrow P$ is its own inverse!

$$P^T P = \underbrace{(I - 2\vec{u}\vec{u}^T)^T}_{P^T}(I - 2\vec{u}\vec{u}^T)$$

$$= (I^T - 2\underbrace{(\vec{u}\vec{u}^T)^T}) (I - 2\vec{u}\vec{u}^T)$$

$$= \underbrace{(I - 2\vec{u}\vec{u}^T)}_{P}(I - 2\vec{u}\vec{u}^T)$$

$$= I - 2\vec{u}\vec{u}^T - 2\vec{u}\vec{u}^T + 4\vec{u}\underbrace{\vec{u}^T\vec{u}}_{=1^2}\vec{u}^T$$

$$= I$$

$$(AB)^T = B^T A^T$$

$$(\vec{u}\vec{u}^T)^T = (\vec{u}^T)^T \vec{u}^T = \vec{u}\vec{u}^T$$

orthogonal matrix $Q \in \mathbb{R}^{n \times n}$

$$\|Q\vec{x}\| = \|\vec{x}\|$$

orthogonal matrices preserve length!

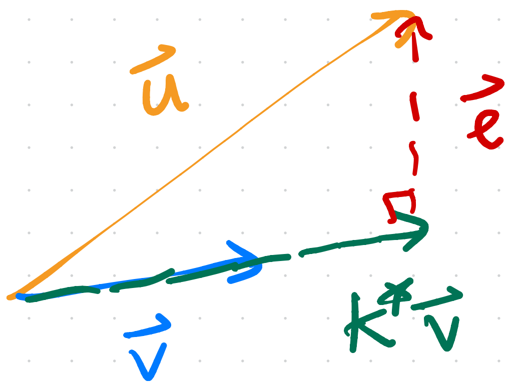
only change direction

$$A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

Projections

"the approximation problem"



choose k^* such that

$$\vec{e} = \vec{u} - k\vec{v}$$

orthogonal to $\vec{v}$

Consider the vectors

$$\vec{x}^{(1)}, \vec{x}^{(2)}, \dots, \vec{x}^{(d)} \in \mathbb{R}^n$$

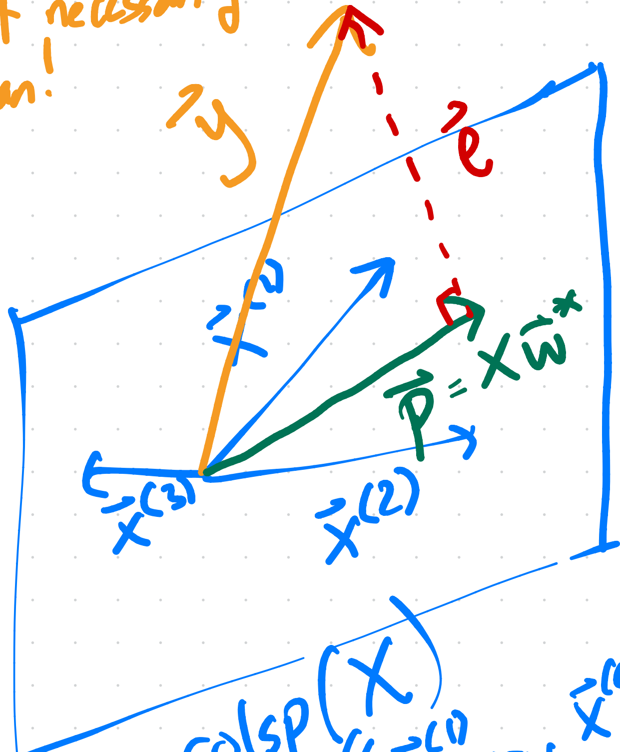
Among all vectors in

$$\underbrace{\text{span}\left(\vec{x}^{(1)}, \dots, \vec{x}^{(d)}\right)}$$

which is closest to $\vec{y}$?

this span is a subspace of $\mathbb{R}^n$

$\vec{y}$ not necessarily in span!



$\text{colsp}(X)$

$= \text{span}\{\vec{x}^{(1)}, \dots, \vec{x}^{(d)}\}$

$=$ a subspace of $\mathbb{R}^n$

$$X = \begin{bmatrix} | & | & & | \\ \vec{x}^{(1)} & \vec{x}^{(2)} & \cdots & \vec{x}^{(d)} \\ | & | & & | \end{bmatrix}_{n \times d}$$

$\vec{x}^{(1)}$ in $\mathbb{R}^n$

vectors in colsp(X) are of the form

$$X\vec{w}, \text{ where } \vec{w} \in \mathbb{R}^d$$

of all $X\vec{w}$'s, which is closest to $\vec{y}$?

choose $\vec{w}^*$ so that

$$\vec{e} = \vec{y} - X\vec{w}^*$$

is orthogonal to

$\text{colsp}(X)$

(equivalent to being orthogonal to each of X's columns) choose $\vec{w}$ so that

$$\vec{x}^{(1)} \cdot (\vec{y} - X\vec{w}) = 0$$

$$\vdots$$

$$\vec{x}^{(d)} \cdot (\vec{y} - X\vec{w}) = 0$$

$$\Downarrow$$

$$X^T(\vec{y} - X\vec{w}) = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$