EECS 245, Winter 2026

LEC 19 Gradient Descent and Convexity

→ Read: Ch. 8

Agenda

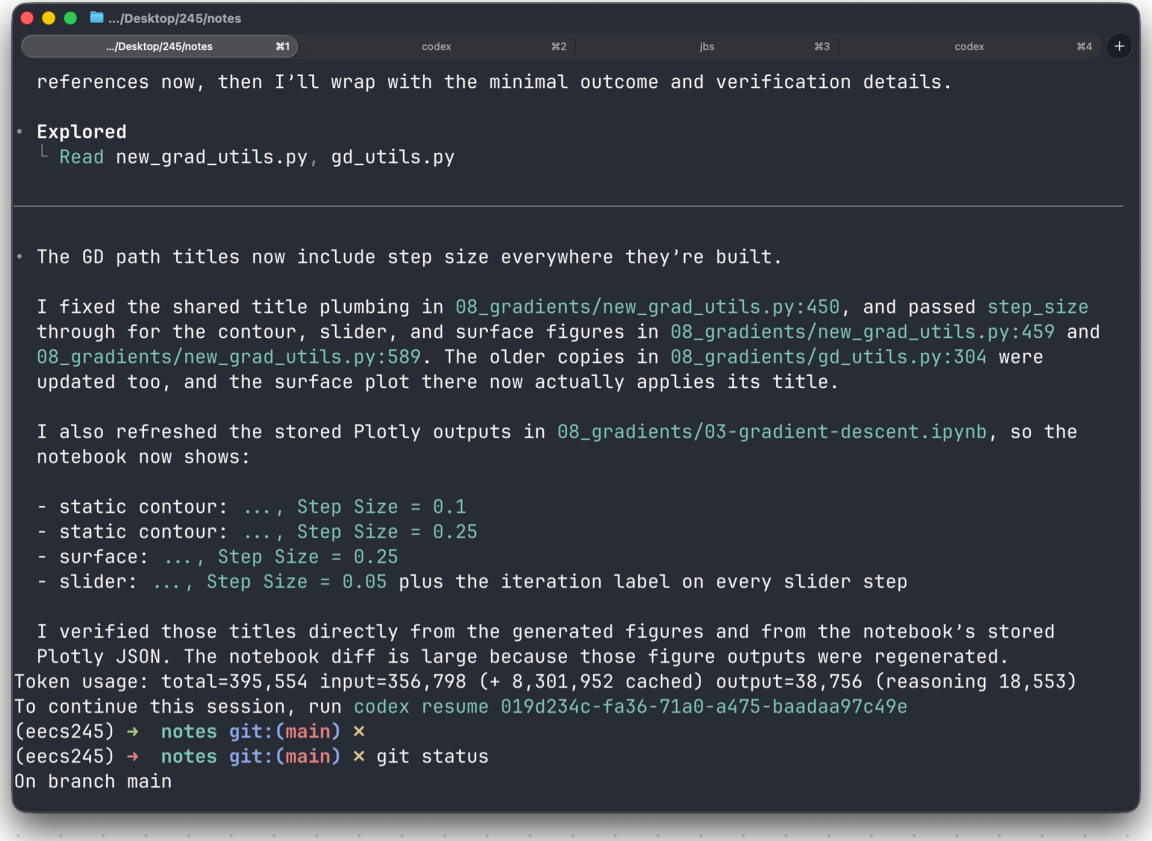
- Gradient descent: what's the point?
  - Update rule
  - Worked example
- Using GD to find optimal model parameters
- When is GD guaranteed to converge?
  - Convexity (not on MT2, but on Final Exam)

Announcements

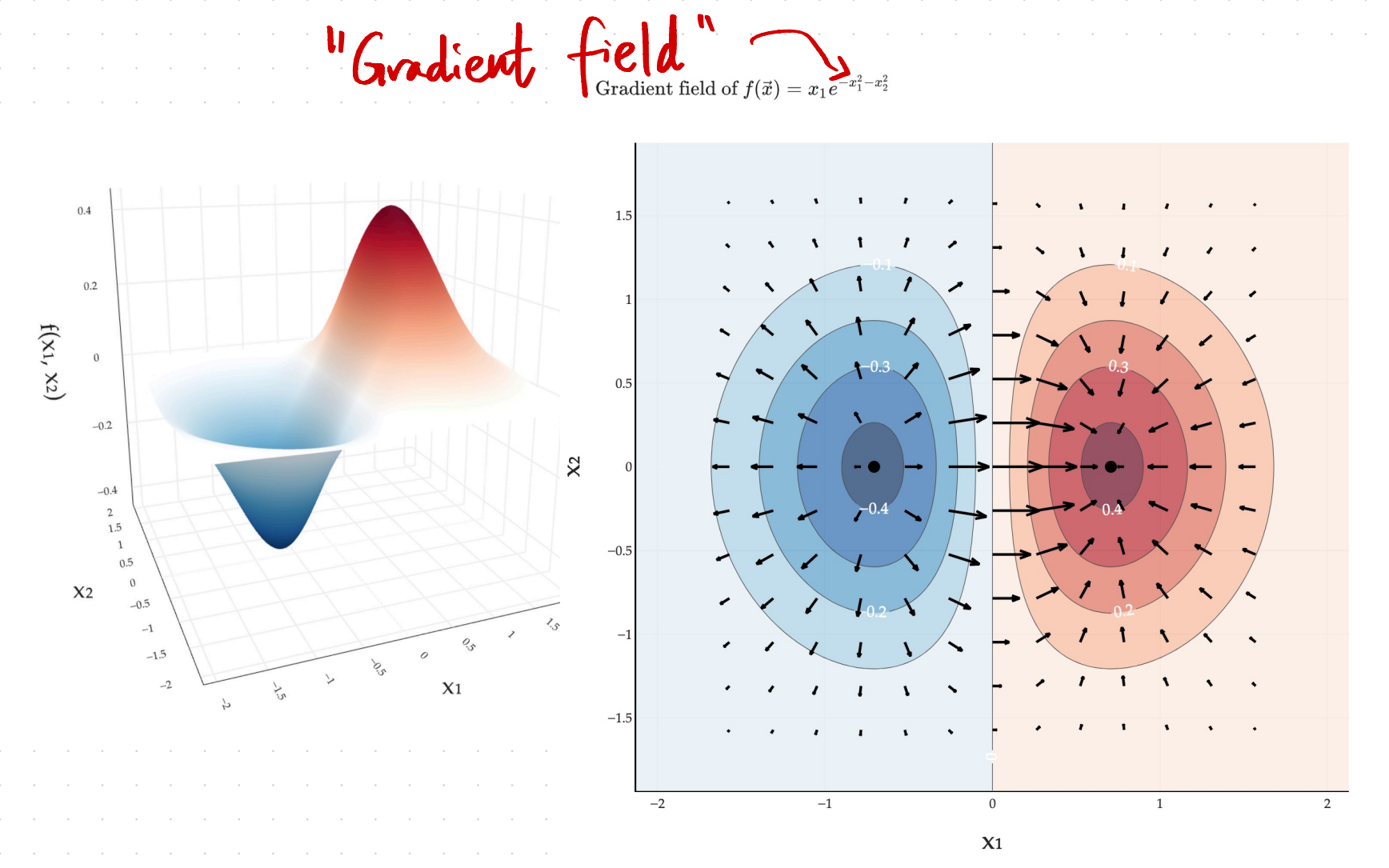
- MT 2 on Monday; see logistics on Ed
- Mock exam Friday: 2:30-5:30PM, 1014 DOW
- HW 9A due Friday, HW 9B out soon
- No in-person lecture next week:
  - office hours Monday
  - Video posted for Weds.

lots of new visuals in Ch. 7 and Ch. 8:

take a look!



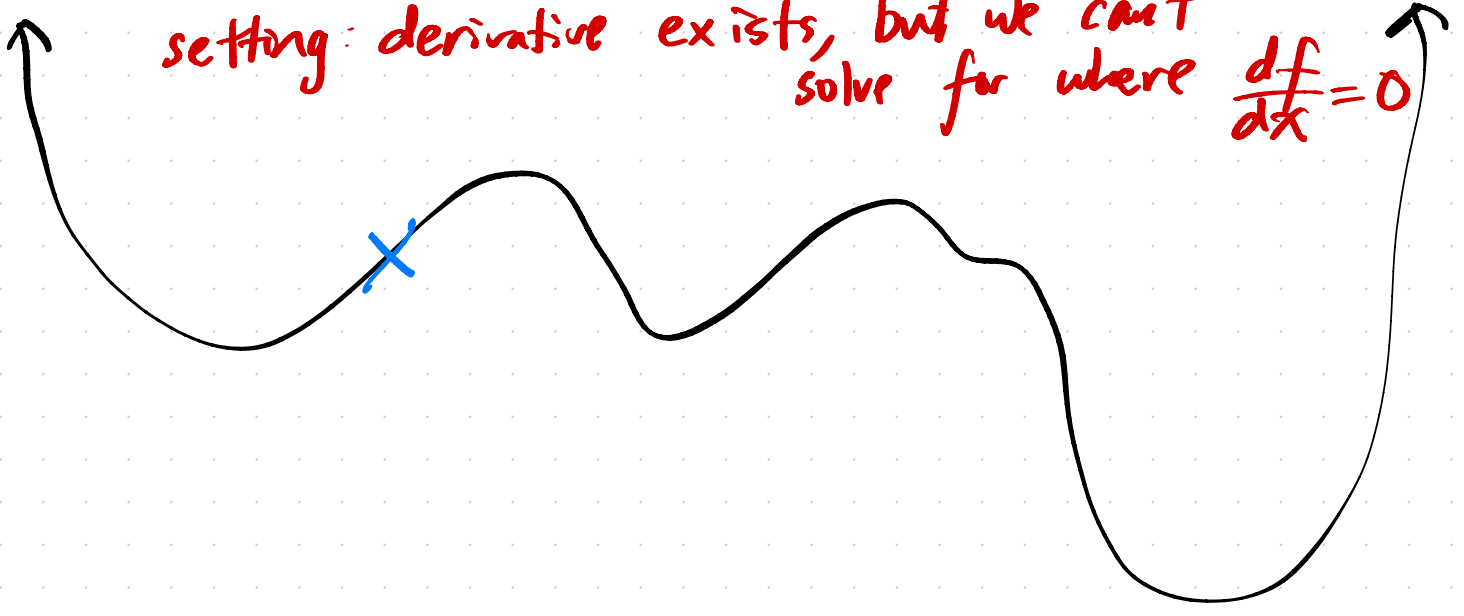
"Gradient field"



What's the point of gradient descent?

⇒ Minimize a function

setting: derivative exists, but we can't solve for where $\frac{df}{dx} = 0$



Gradient descent

Goal: Find the $\vec{x}^*$ that minimizes $f(\vec{x})$

Given an initial guess, $\vec{x}^{(0)}$

and a step size, $\alpha > 0$ (also called learning rate)

Update rule:

$$\vec{x}^{(t+1)} = \vec{x}^{(t)} - \alpha \nabla f(\vec{x}^{(t)})$$

- $\vec{x}^{(t+1)}$: guess at iteration $t+1$
- $-\nabla f(\vec{x}^{(t)})$: opposite direction from gradient
- $\nabla f(\vec{x}^{(t)})$: gradient at old guess
- α: scalar

Terminate when $\| \nabla f(\vec{x}^{(t)}) \| \leq 0.001$ (tolerance)

Activity 1, 8.3

$$f(\vec{x}) = (x_1 - 2)^2 + 2x_1 - (x_2 - 3)^2$$

$\vec{x} \in \mathbb{R}^2$

Given $\vec{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\alpha = \frac{1}{3}$

Find $\vec{x}^{(1)} = \vec{x}^{(0)} - \alpha \nabla f(\vec{x}^{(0)})$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2 \end{bmatrix}$$

$$f(\vec{x}) = (x_1 - 2)^2 + 2x_1 - (x_2 - 3)^2$$

$$\nabla f(\vec{x}) = \begin{bmatrix} 2(x_1 - 2) + 2 \\ -2(x_2 - 3) \end{bmatrix}$$

at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\nabla f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2(0-2)+2 \\ -2(0-3) \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

Gradient descent is made for minimizing average loss to find optimal model parameters

$$R_{sq}(\vec{w}) = \frac{1}{n} \|\vec{y} - X\vec{w}\|^2$$

- don't need gradient descent to find $\vec{w}^*$: there is a closed form sol'n,

$$\vec{w}^* = (X^T X)^{-1} X^T \vec{y}$$

- GD could be quicker if the data is massive

what if we used GD to minimize

$$R_{sq}(\vec{w}) = \frac{1}{n} \|\vec{y} - X\vec{w}\|^2 \quad ?$$

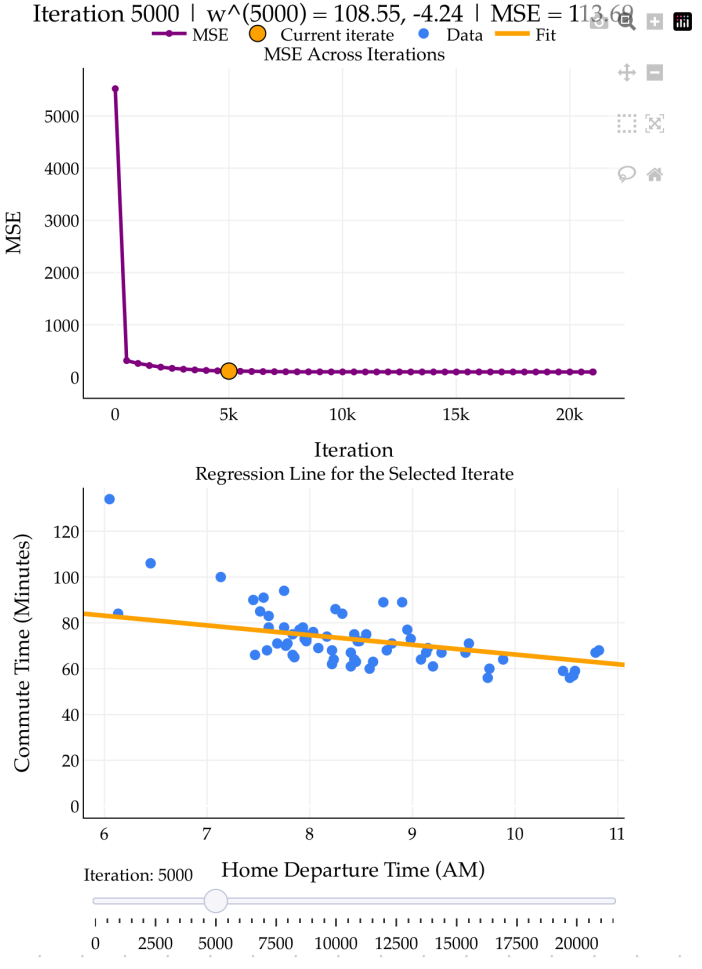
$$\nabla R_{sq}(\vec{w}) = \frac{-2}{n} \left(X^T \vec{y} - X^T X \vec{w} \right)$$

$$\vec{w}^{(t+1)} = \vec{w}^{(t)} - \alpha \left(\frac{-2}{n} X^T \vec{y} - X^T X \vec{w}^{(t)} \right)$$

let's visualize: look at 8.4!

```
i =     0 | w^(i) = [   0.000,    0.000] | mse = 5523.5231 | ||grad|| = 1229.842640
i =   500 | w^(i) = [  19.804,    6.102] | mse =  314.8511 | ||grad|| =  3.527944
i =  1000 | w^(i) = [  36.133,    4.200] | mse =  260.7135 | ||grad|| =  3.058219
i =  1500 | w^(i) = [  50.289,    2.551] | mse =  220.0323 | ||grad|| =  2.651035
i =  2000 | w^(i) = [  62.559,    1.121] | mse =  189.4630 | ||grad|| =  2.298065
i =  2500 | w^(i) = [  73.196,   -0.118] | mse =  166.4919 | ||grad|| =  1.992091
i =  3000 | w^(i) = [  82.416,   -1.193] | mse =  149.2306 | ||grad|| =  1.726856
i =  3500 | w^(i) = [  90.409,   -2.124] | mse =  136.2598 | ||grad|| =  1.496935
i =  4000 | w^(i) = [  97.338,   -2.931] | mse =  126.5130 | ||grad|| =  1.297627
i =  4500 | w^(i) = [ 103.344,   -3.631] | mse =  119.1889 | ||grad|| =  1.124856
i =  5000 | w^(i) = [ 108.551,   -4.237] | mse =  113.6852 | ||grad|| =  0.975088
i =  5500 | w^(i) = [ 113.064,   -4.763] | mse =  109.5496 | ||grad|| =  0.845261
i =  6000 | w^(i) = [ 116.976,   -5.219] | mse =  106.4419 | ||grad|| =  0.732719
i =  6500 | w^(i) = [ 120.368,   -5.614] | mse =  104.1067 | ||grad|| =  0.635162
i =  7000 | w^(i) = [ 123.308,   -5.957] | mse =  102.3519 | ||grad|| =  0.550594
i =  7500 | w^(i) = [ 125.856,   -6.254] | mse =  101.0333 | ||grad|| =  0.477285
i =  8000 | w^(i) = [ 128.065,   -6.511] | mse =  100.0424 | ||grad|| =  0.413738
i =  8500 | w^(i) = [ 129.980,   -6.734] | mse =   99.2978 | ||grad|| =  0.358651
i =  9000 | w^(i) = [ 131.640,   -6.928] | mse =   98.7383 | ||grad|| =  0.310899
i =  9500 | w^(i) = [ 133.079,   -7.095] | mse =   98.3179 | ||grad|| =  0.269504
i = 10000 | w^(i) = [ 134.327,   -7.241] | mse =   98.0020 | ||grad|| =  0.233621
i = 10500 | w^(i) = [ 135.408,   -7.367] | mse =   97.7646 | ||grad|| =  0.202516
i = 11000 | w^(i) = [ 136.345,   -7.476] | mse =   97.5862 | ||grad|| =  0.175552
i = 11500 | w^(i) = [ 137.158,   -7.571] | mse =   97.4521 | ||grad|| =  0.152179
i = 12000 | w^(i) = [ 137.862,   -7.653] | mse =   97.3514 | ||grad|| =  0.131917
i = 12500 | w^(i) = [ 138.473,   -7.724] | mse =   97.2757 | ||grad|| =  0.114353
```

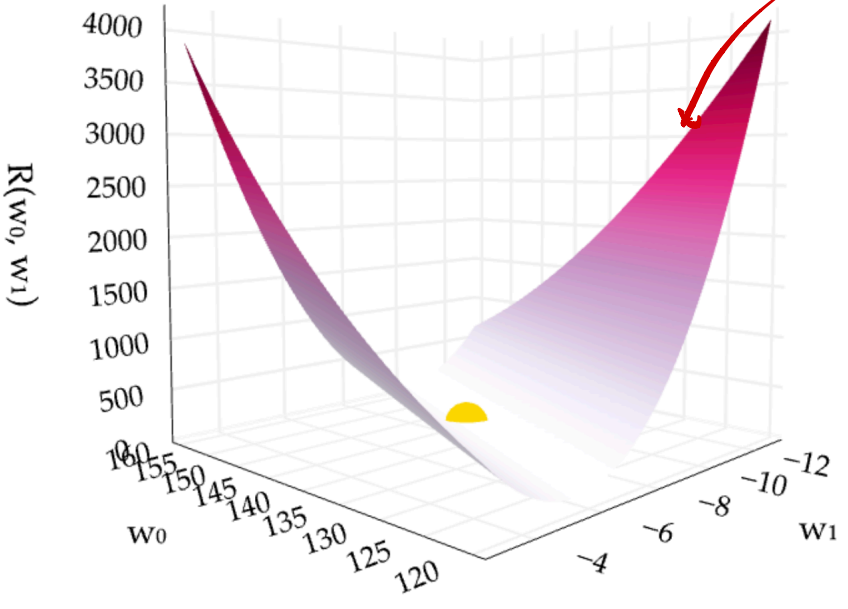
approaching 0



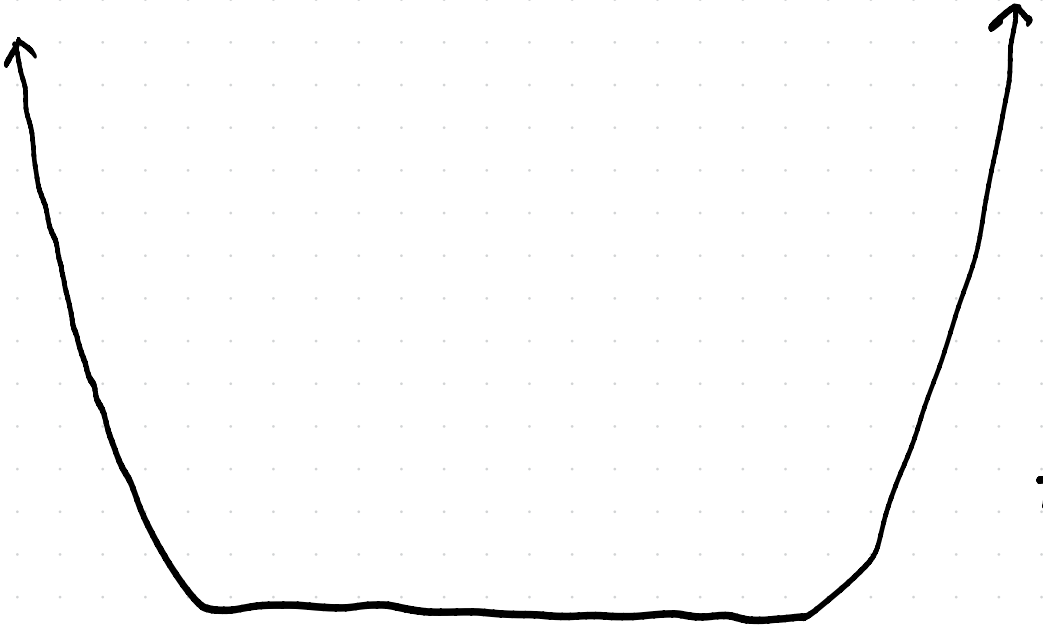
as t increases,

MSE approaches

theoretical minimum



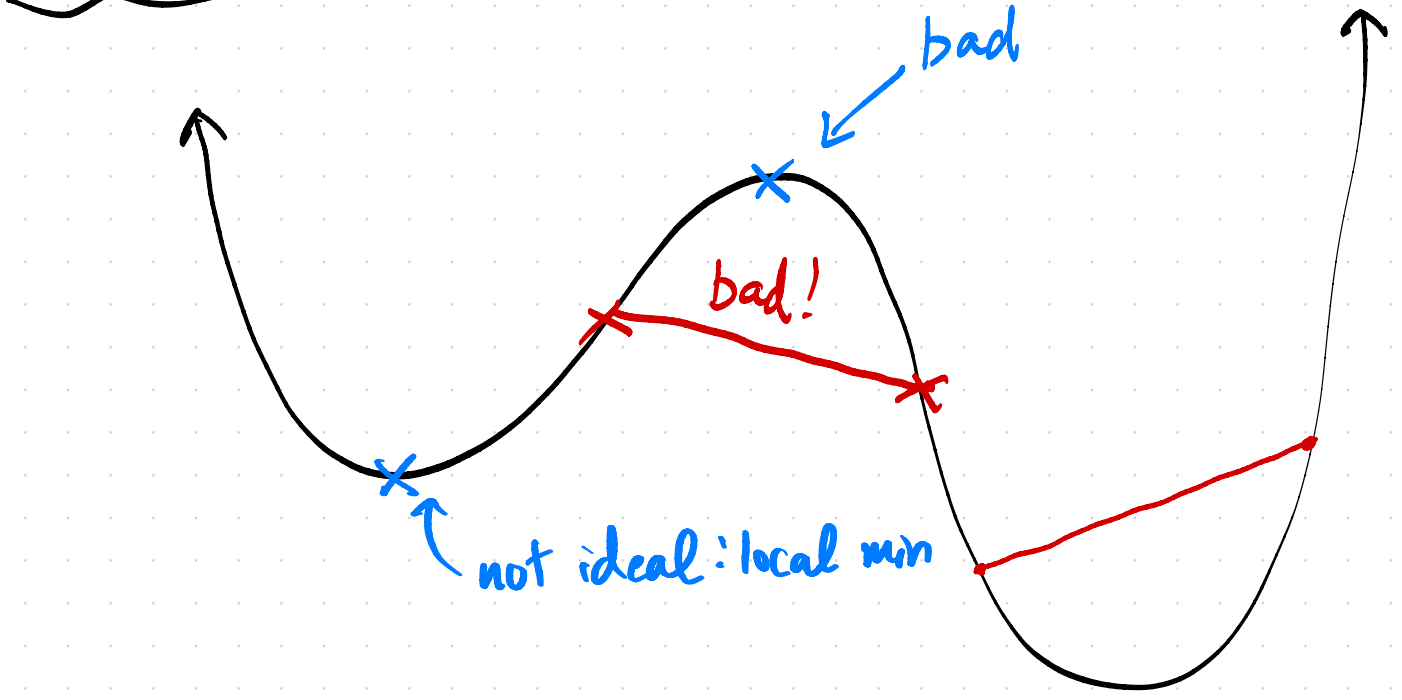
this is the function we just used GD to minimize!

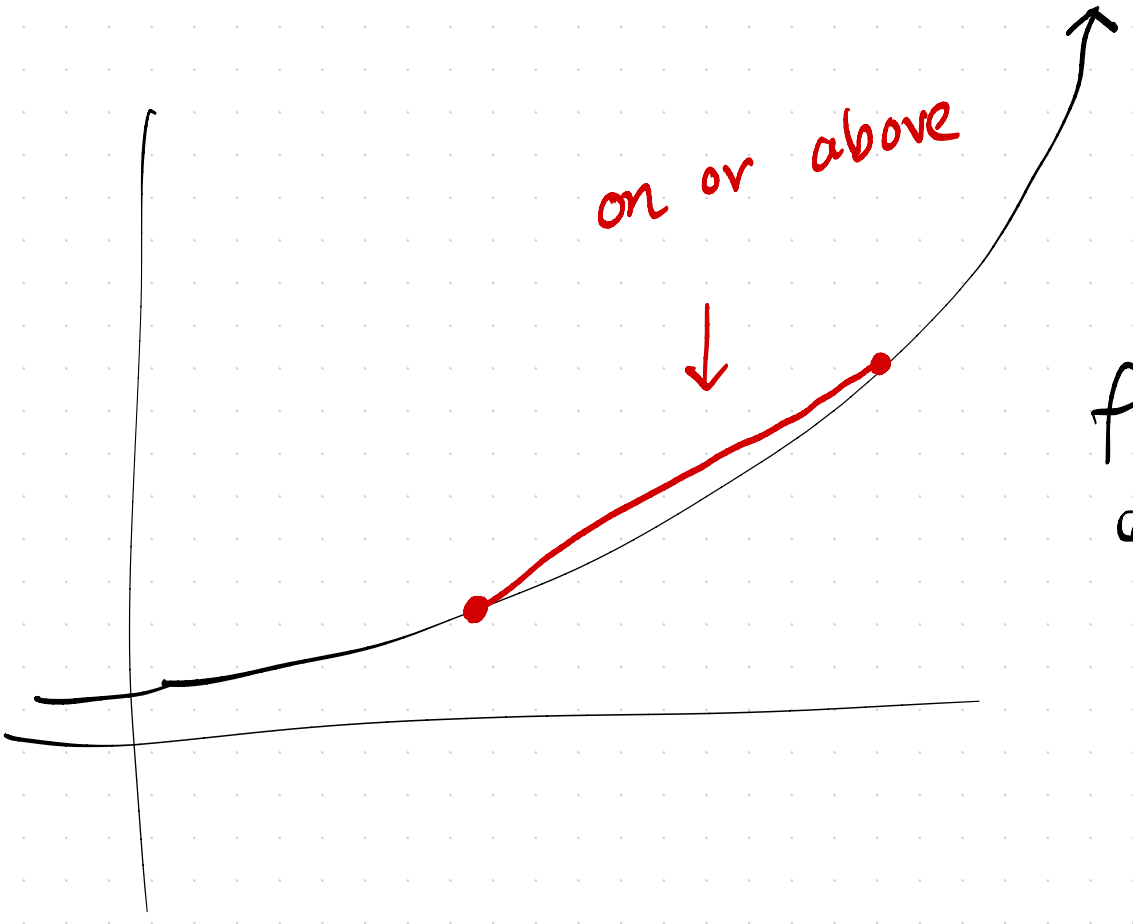


when would $R_{sq}(\vec{w})$ look like this?

→ if X's columns are linearly dependent

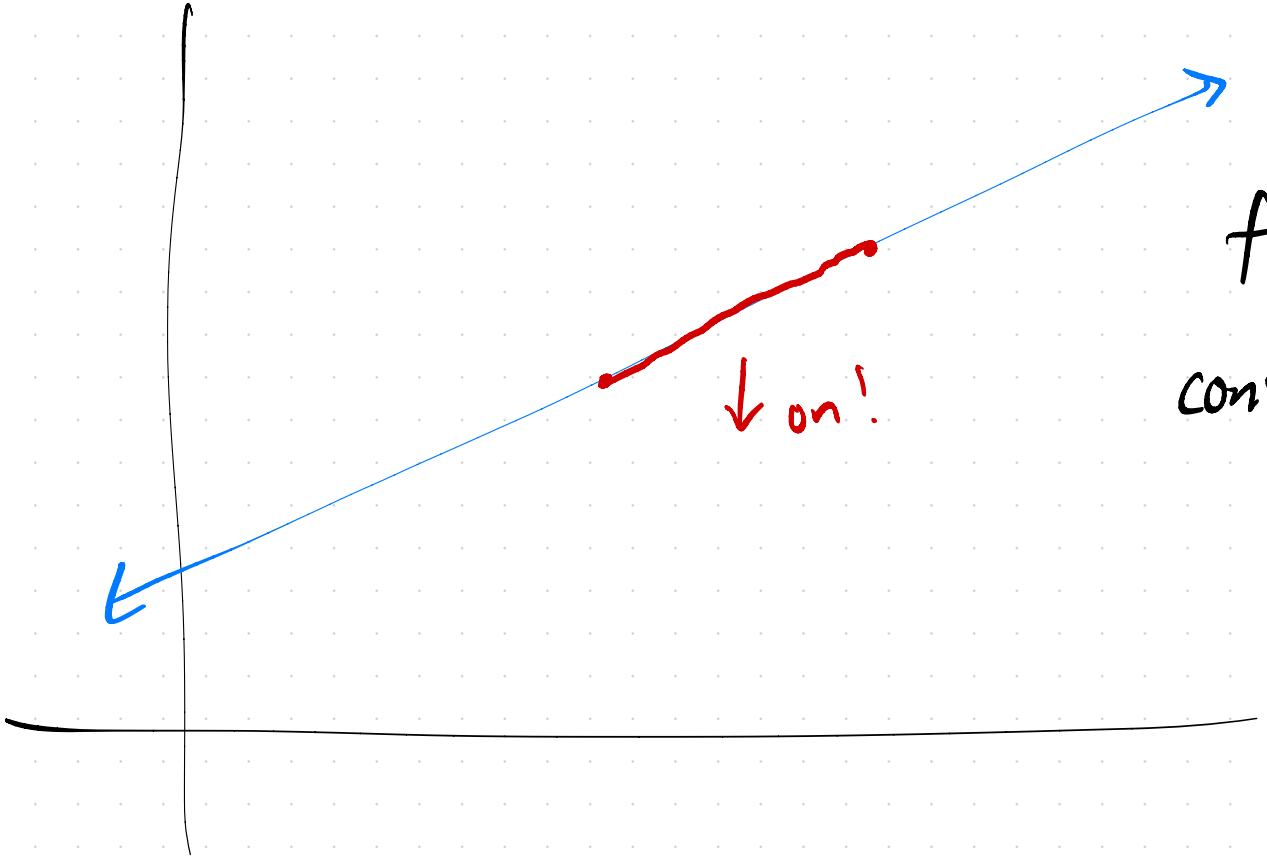
Is this function convex? no!





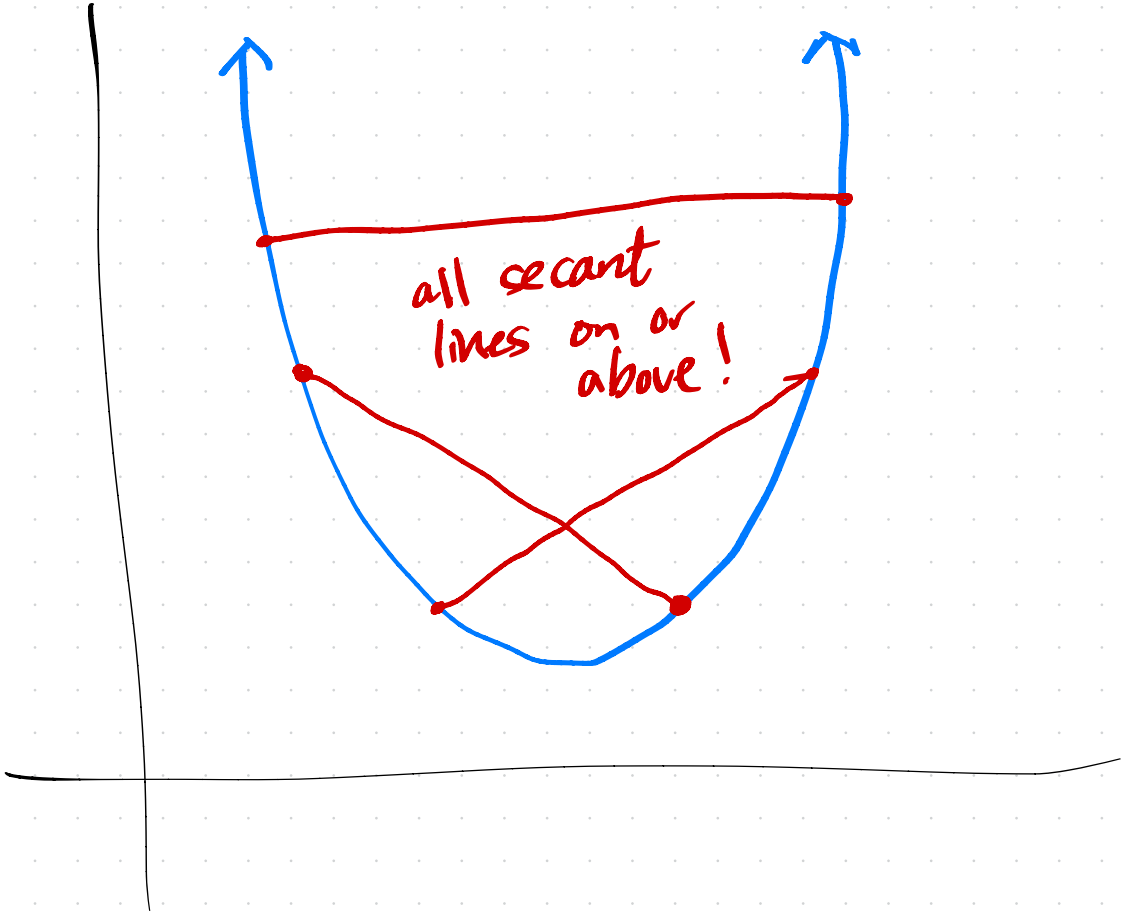
$f(x) = e^x$

convex? yes!



$f(x) = 2x - 3$

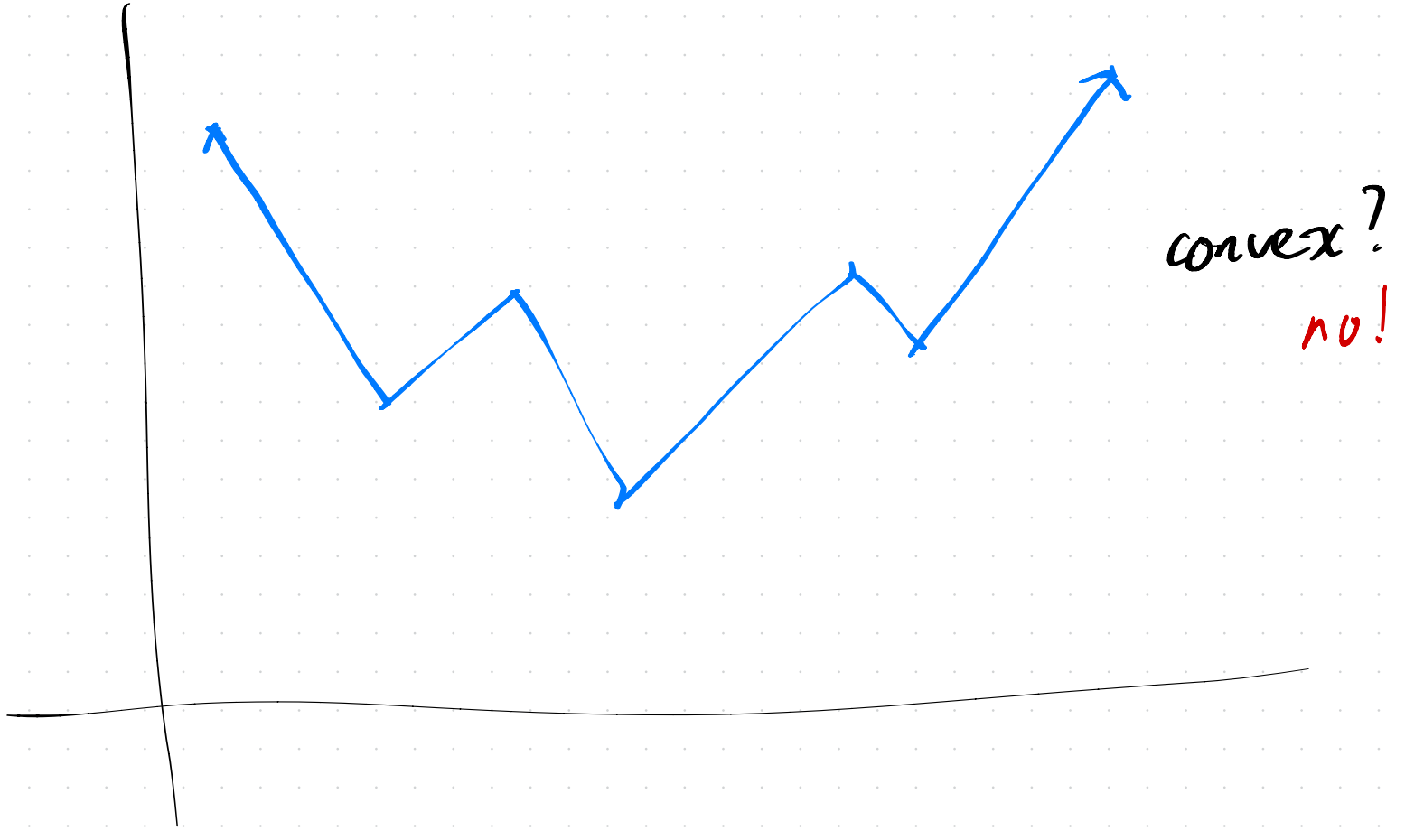
convex? yes!



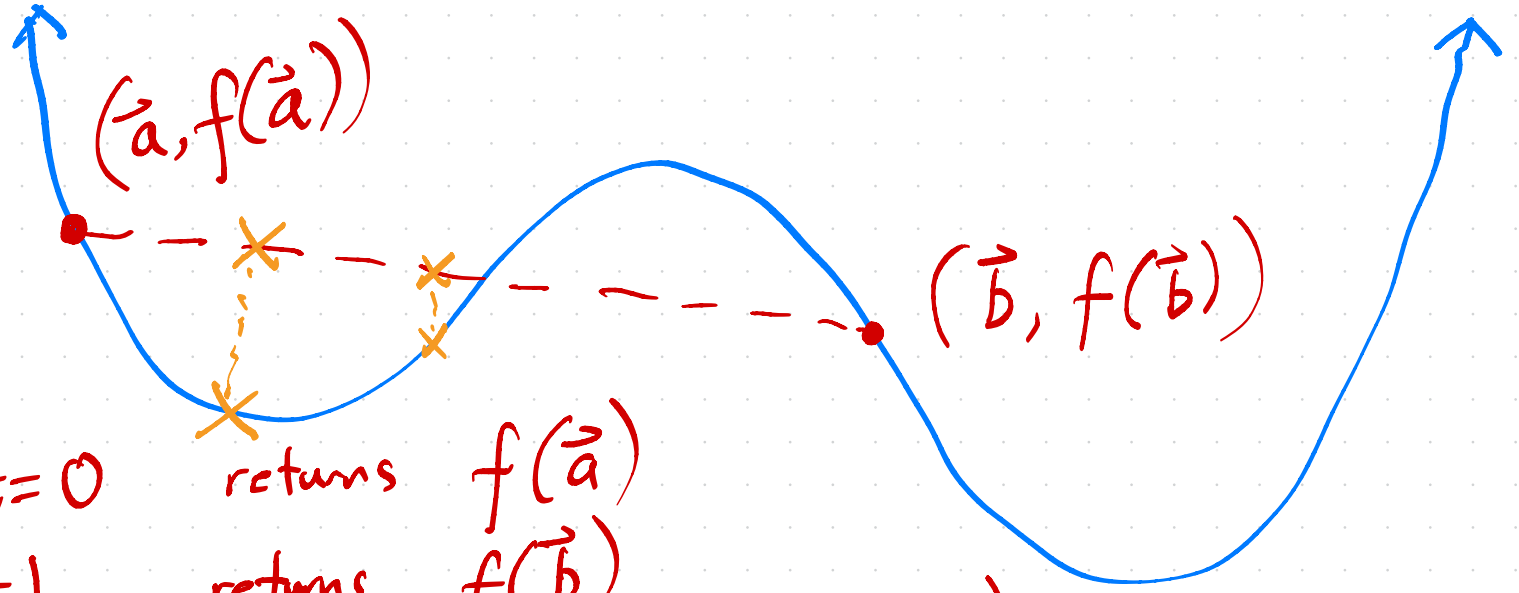
$f(x) = x^2 + 7$

convex?

yes!


convex?
no!

Idea: f is convex if every possible secant line is on or above the curve



$t=0$ returns $f(\vec{a})$

$t=1$ returns $f(\vec{b})$

$$L(t) = f(\vec{a}) + t\left(f(\vec{b}) - f(\vec{a})\right)$$

$$= \underbrace{(1-t)}\, f(\vec{a}) + \underbrace{t}\, f(\vec{b})$$

$f: \mathbb{R}^d \to \mathbb{R}$

"Formal definition"

f is convex if for all $\vec{a}, \vec{b}$ in its domain, and any $t \in [0,1]$,

$$f\left((1-t)\vec{a} + t\vec{b}\right) \leq (1-t)f(\vec{a}) + tf(\vec{b})$$

point on function t% of way from $\vec{a}$ to $\vec{b}$

point on line t% of way from $(\vec{a}, f(\vec{a}))$, $(\vec{b}, f(\vec{b}))$

This definition of convexity doesn't require derivatives; second derivative test for vector-to-scalar functions is more complicated

Why do we care? Convex functions play nice with gradient descent: no "traps" (local minima that aren't global minima)