

$$X^T X + \lambda I.$$

why is this always invertible?

square, $n \times n$ matrix A

think about linear transformations
from $\mathbb{R}^n \rightarrow \mathbb{R}^n$

(non-zero)
eigenvector \vec{v} :

$$A\vec{v} = \lambda\vec{v}$$

when multiplied by A , \vec{v} 's direction
didn't change; it was just
scaled by a factor of

eigenvalue

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

if \vec{v} is an eigenvector of A ,
so is $C\vec{v}$, for any $C \neq 0$

Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A\vec{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ eigenvector of A
with $\lambda_1 = 3$

$$A \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

$$= 3 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \text{so } \vec{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

is also

eigvec of A
with $\lambda = 3$

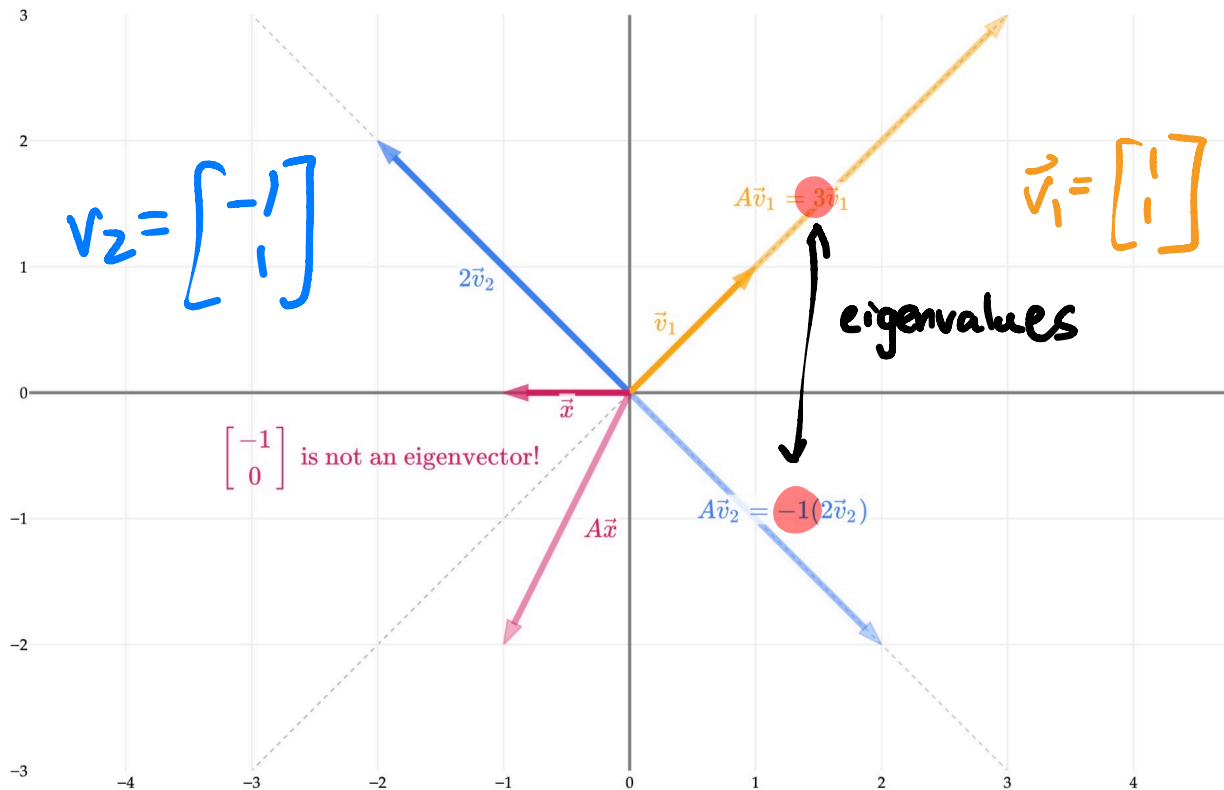
$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+2 \\ -2+1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

so, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector

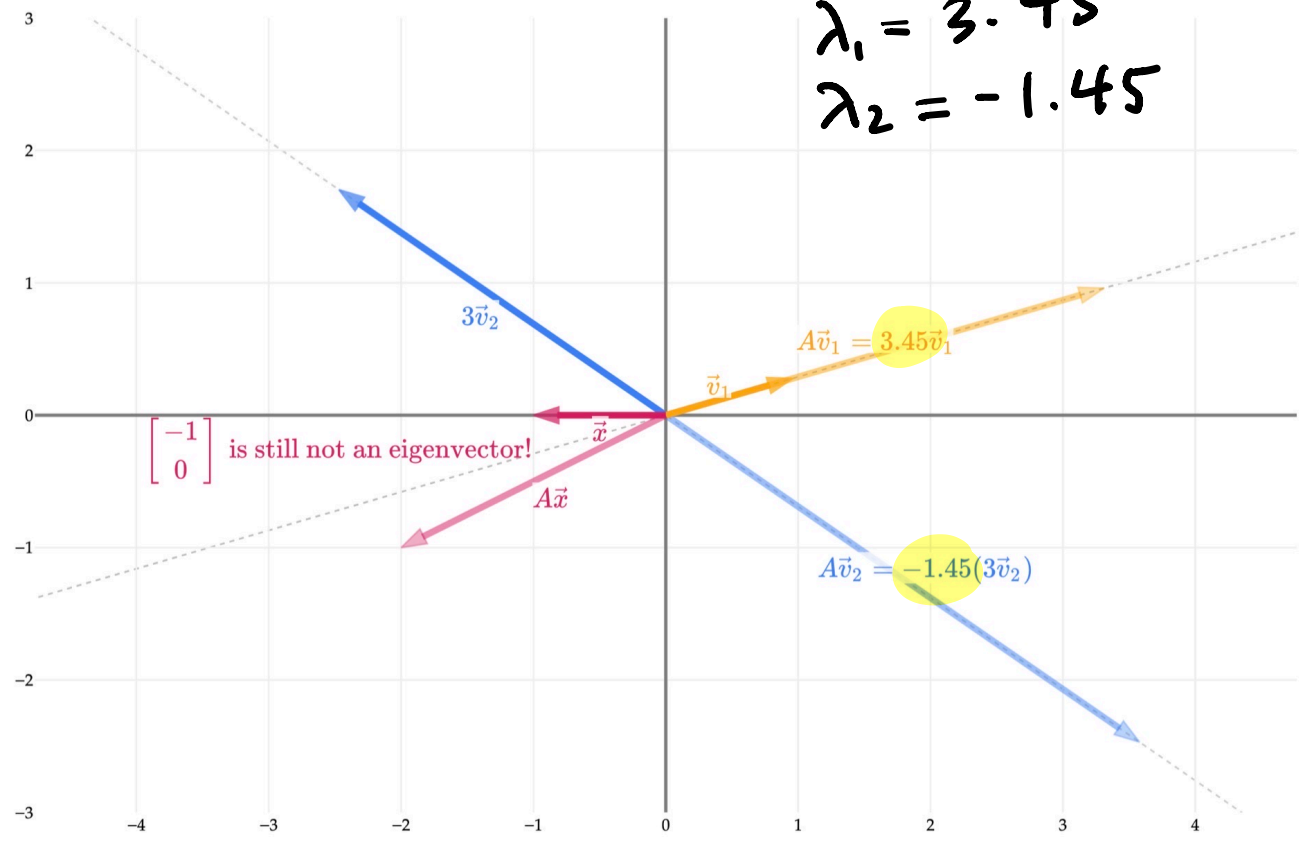
corresponding to $\lambda_2 = -1$

Visualizing the eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$



Visualizing the eigenvectors of $B = \begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix}$

Here,
 $\lambda_1 = 3.45$
 $\lambda_2 = -1.45$



```
B = np.array([[2, 5],  
              [1, 0]])
```

```
np.linalg.eig(B)
```

```
EigResult(eigenvalues=array([ 3.44948974, -1.44948974]), eigenvectors=array([[  
0.96045535, -0.82311938],  
 [ 0.27843404,  0.56786837]]))
```

cols are the
eigvecs

\vec{v}_1

\vec{v}_2

observe: $\lambda_1 + \lambda_2 = 2$,

which is sum of diagonal!

True in general: sum of λ_i 's = sum of diagonal

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix}$$

notice: $\text{rank}(A) = 1$

$$\lambda_1 = 13, \lambda_2 = \underline{0}?$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

↑
basis for $\text{nullsp}(A)$!

$$\lambda_1 = 13 \quad \begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13x \\ 13y \end{bmatrix}$$

$$x + 4y = 13x \Rightarrow 4y = 12x \Rightarrow y = 3x$$

$$3x + 12y = 13y$$

$$\Downarrow 3x = y$$

both same,
since any vector
on that line
works!

e.g. $x=1$, so $y=3$,

so $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

A has an
eigenvalue of 0



A is not invertible

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ had}$$

$$\lambda_1 = 3 \text{ with } \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1 \text{ with } \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

what are the
eigenvalues of
 A^2 ?

what eigenvectors?

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \\ = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

if \vec{v} is an eigvec with eigval λ
of A , then

$$\begin{aligned} A^2 \vec{v} &= AA\vec{v} = A(\lambda\vec{v}) = \lambda(A\vec{v}) \\ &= \lambda(\lambda\vec{v}) \\ &= \lambda^2 \vec{v} \end{aligned}$$

only one direction!

then, \vec{v} is an eigvec
of A^2 with eigval λ^2 .

possible

for A^2 to have an
eigenvector

that A doesn't have

think: rotations

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

identity
matrix



\vec{v} is in $A - \lambda I$'s null space

$\Rightarrow A - \lambda I$ has a non-trivial null space

$\Rightarrow \det(A - \lambda I) = 0 \rightarrow$ use this to find λ 's!

characteristic
polynomial

$$p(\lambda) = \det(A - \lambda I)$$

degree n
polynomial

eigenvalues are solutions to

$$p(\lambda) = \det(A - \lambda I) = 0$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \left(\begin{bmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{bmatrix} \right)$$

$$= (1 - \lambda)^2 - 4$$

$$= \lambda^2 - 2\lambda + 1 - 4$$

$$= \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0$$

$$\lambda = -1, \lambda = 3$$

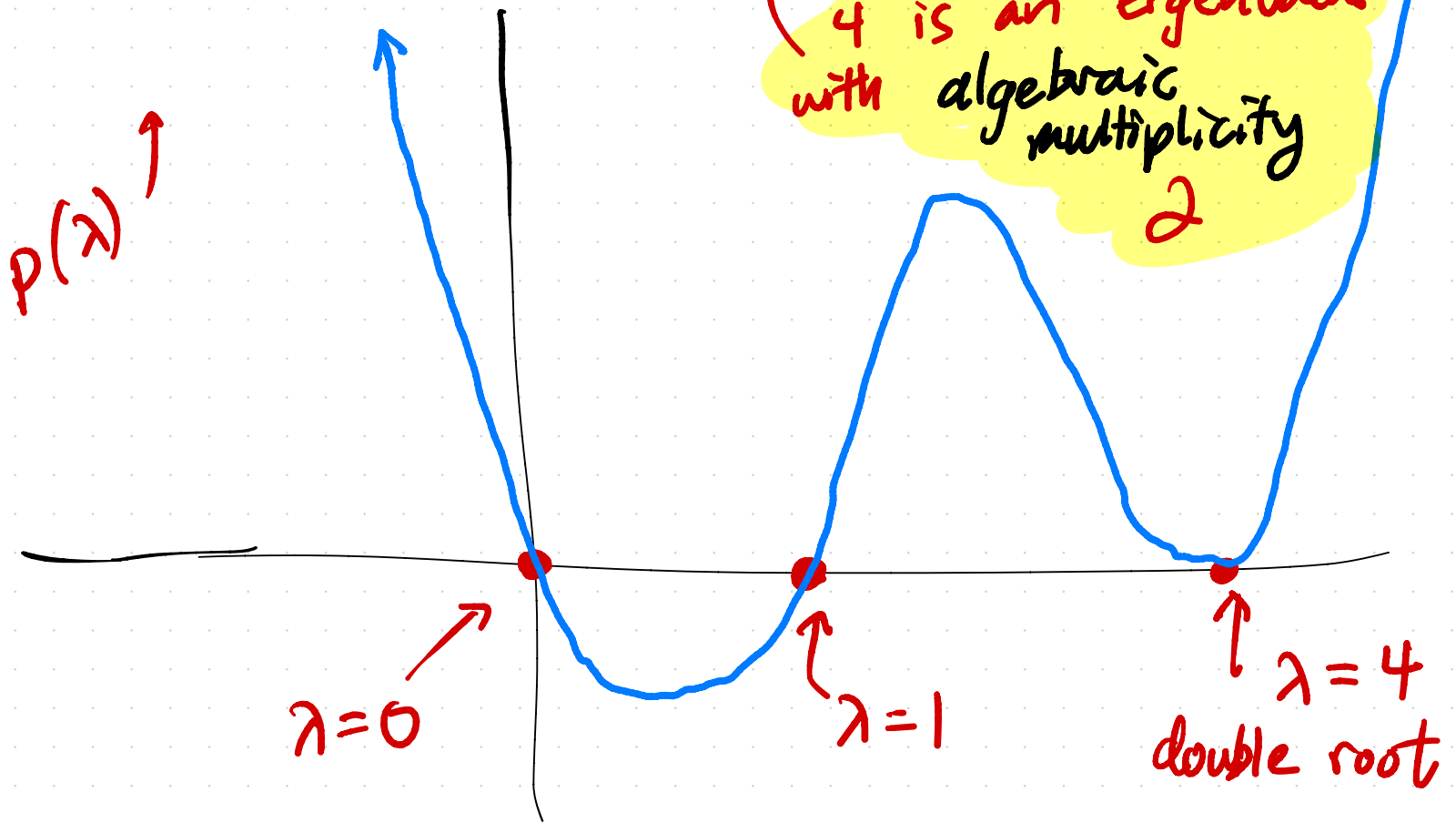
$$A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 4-\lambda & 0 \\ 0 & 4-\lambda \end{pmatrix}$$

$$= (4-\lambda)(4-\lambda)$$

$$= (\lambda-4)^2$$

$$p(\lambda) = \lambda(\lambda-1)(\lambda-4)^2$$

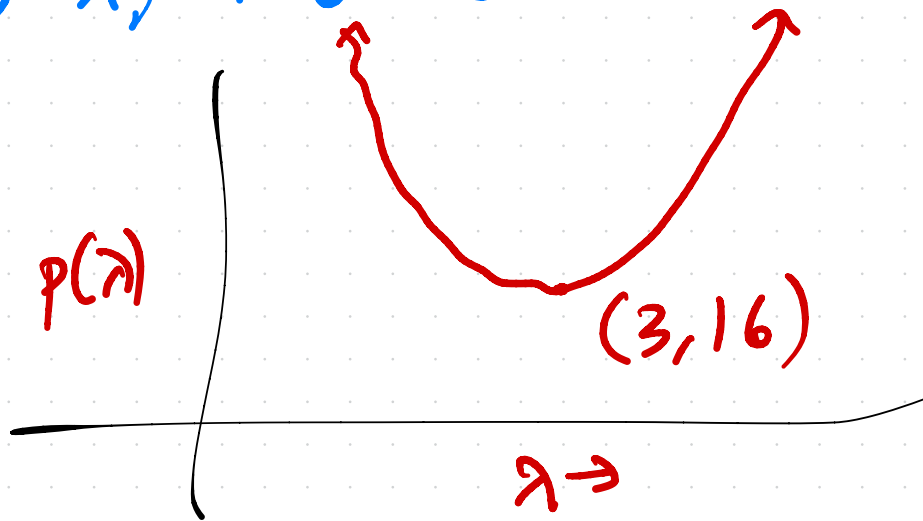


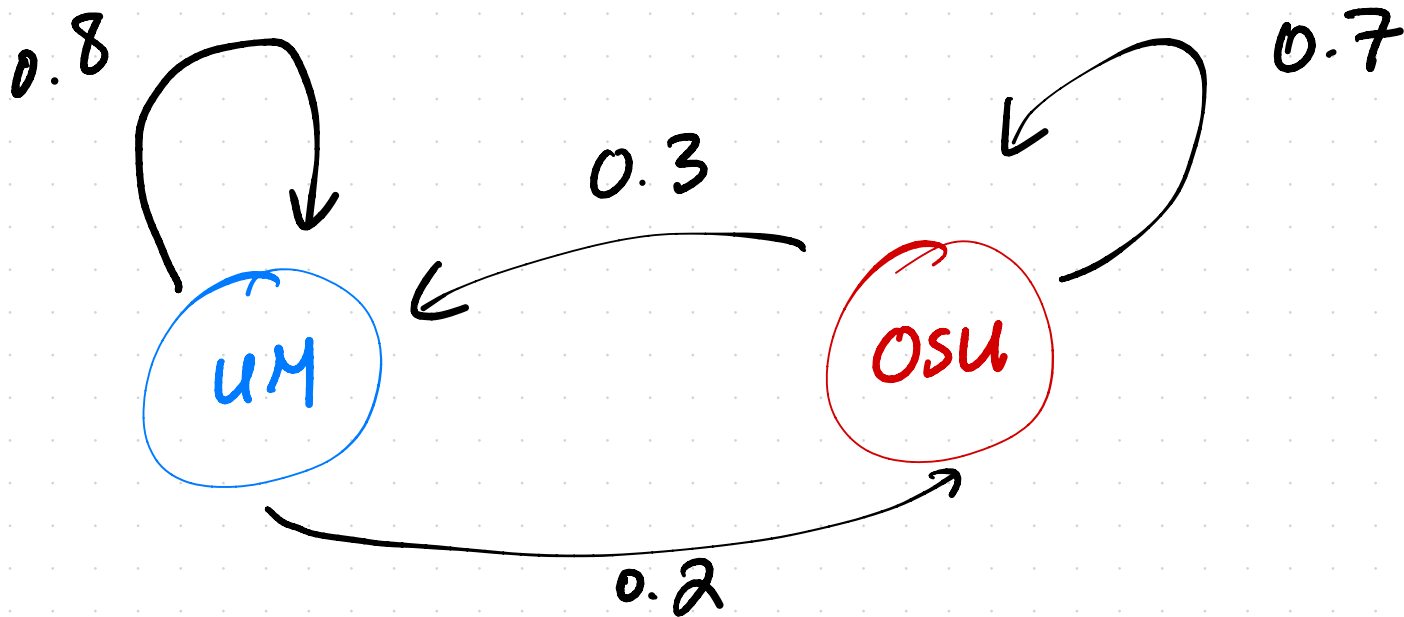
$$A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} = 5 \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

$$5e^{i\theta} \quad 5e^{-i\theta}$$
$$\theta = \cos^{-1}(3/5)$$

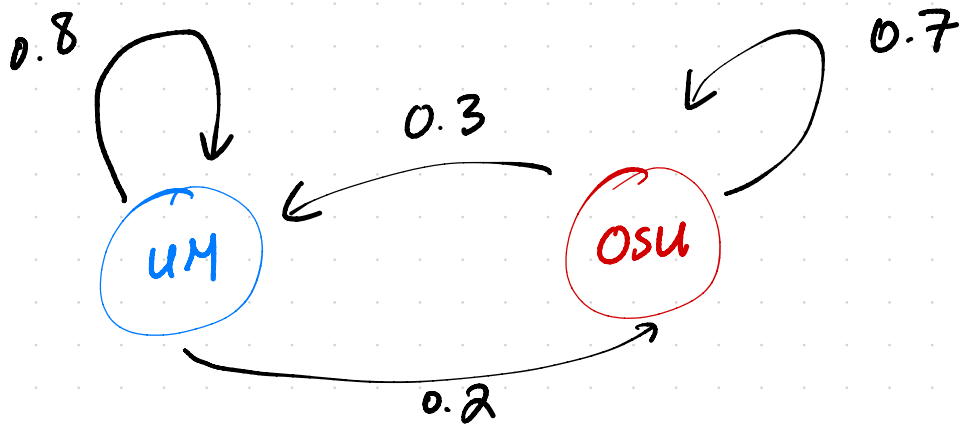
$$p(\lambda) = (3 - \lambda)^2 + 16 = 0$$

rotation, then stretch





in the long run, what %
of games does Michigan win?



$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{array}{l} \rightarrow \text{UM} \\ \rightarrow \text{OSU} \end{array}$$

UM \rightarrow OSU \rightarrow

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

$$\vec{x}_k = \begin{bmatrix} p(\text{Michigan})_k \\ p(\text{OSU})_k \end{bmatrix}$$

simulate!

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}_1 = A \vec{x}_0 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

$$\vec{x}_2 = A \vec{x}_1 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = A^2 \vec{x}_0$$

.....
in general,

$$\vec{x}_k = A^k \vec{x}_0$$

multiplying by A steps

one iteration

into the future!

... big idea

(see notes for code):

```
x_1 = [0.8 0.2]
x_2 = [0.7 0.3]
x_3 = [0.65 0.35]
x_4 = [0.625 0.375]
x_5 = [0.6125 0.3875]
x_6 = [0.60625 0.39375]
x_7 = [0.603125 0.396875]
x_8 = [0.6015625 0.3984375]
x_9 = [0.60078125 0.39921875]
x_10 = [0.60039063 0.39960938]
x_11 = [0.60019531 0.39980469]
x_12 = [0.60009766 0.39990234]
x_13 = [0.60004883 0.39995117]
x_14 = [0.60002441 0.39997559]
```

$\vec{x}_k \rightarrow$ an eigen
vec
of A

(corresponding
to $\lambda = 1$)

$$A \vec{x} = \underline{1} \vec{x}$$

$$\vec{x} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$0.8x_1 + 0.3x_2 = x_1$$

$$0.2x_1 + 0.7x_2 = x_2$$

$$\vec{x} = c \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

add condⁿ
that
 $x_1 + x_2 = 1$