EECS 245, Winter 2026

LEC 21

Adjacency Matrices;
Linearly Independent
Eigenvectors

Going to assume you watched the videos
for Lecture 20!

Agenda

Ch. 9.1-9.2

- Recap: Eigenvalues and eigenvectors
- Recap: Adjacency matrices 9.3
- The "dominant eigenvalue" 9.3
- What if an $n \times n$ matrix has n linearly independent eigenvectors?
  - A is diagonalizable
  - Related: algebraic and geometric multiplicities

Announcements

- HW 9B due tomorrow
- HW 10 + 11 will be due on Tuesdays
- MT 2 scores on Gradescope
  - I have office hours every day this week
- Watch videos for Lecture 20!

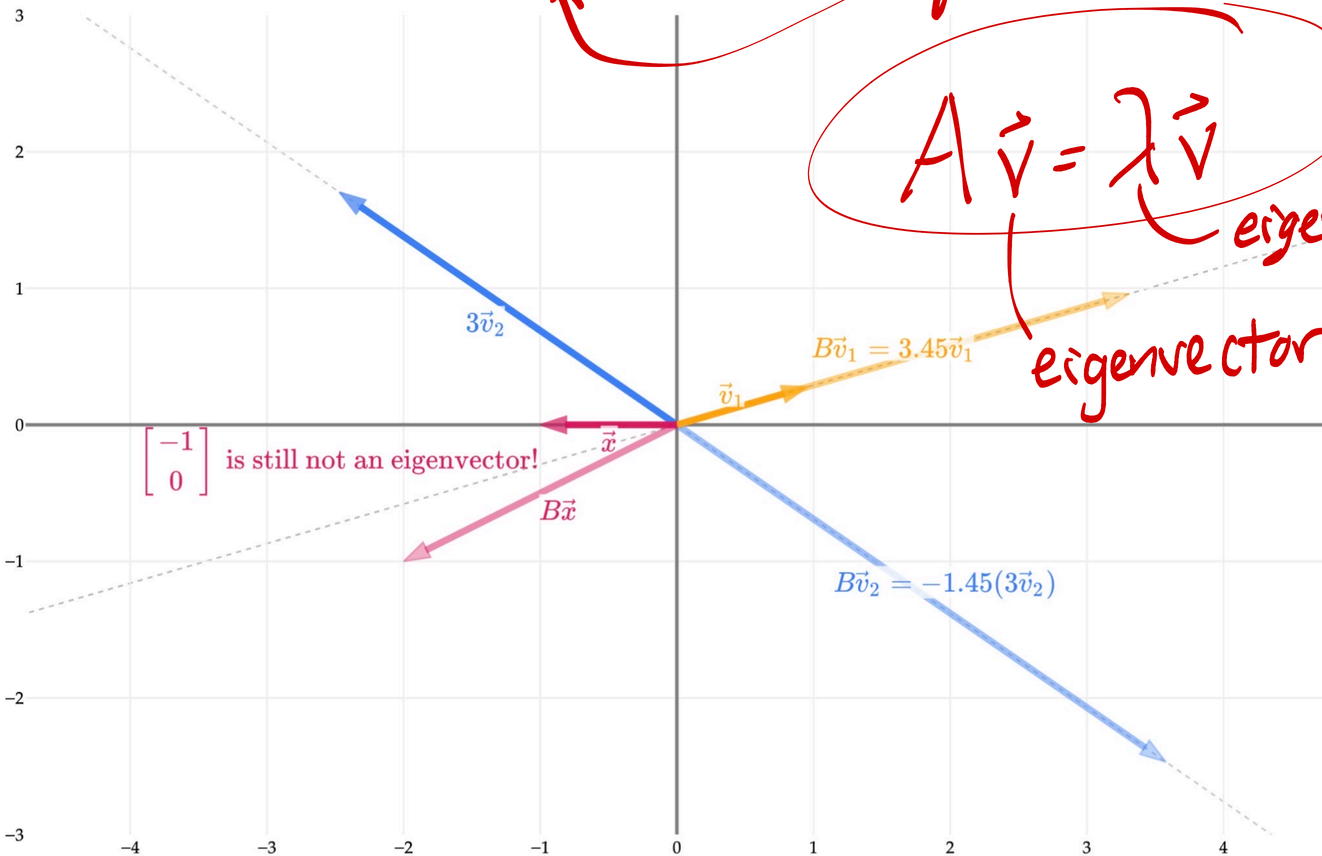
Visualizing the eigenvectors of $B = \begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix}$

square matrix

$$A\vec{v} = \lambda \vec{v}$$

eigenvalue

eigenvector



Example (Ch. 9.2):

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

Find A's eigenvalues and eigenvectors.

Quick way to find λ's:

$$\lambda_1 + \lambda_2 = 7 \quad \text{trace (sum of diag)}$$

$$\lambda_1 \lambda_2 = 10$$

$$\Rightarrow \lambda_1 = 5, \ \lambda_2 = 2$$

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$p(\lambda) = \det(A - \lambda I)$$

$$= \begin{vmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix}$$

$$= (3-\lambda)(4-\lambda) - 2$$

trace

$$= \lambda^2 - 7\lambda + 12 - 2 = \lambda^2 - 7\lambda + 10$$

determinant

$$\lambda^2 - 7\lambda + 10 = 0$$

$$(\lambda - 2)(\lambda - 5) = 0$$

$$\downarrow \qquad \downarrow$$

$$\lambda = 2 \text{ or } 5$$

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\lambda_1 = 5$$

Let $\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$

$$A\vec{v}_1 = 5\vec{v}_1$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5a \\ 5b \end{bmatrix}$$

$$3a + b = 5a \Rightarrow 2a = b$$

$$2a + 4b = 5b \Rightarrow 2a = b$$

as we expect, there are infinitely many a's and b's!

pick $a = 1$

$b = 2$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\lambda_2 = 2$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$$

$$3a + b = 2a$$

$$b = -a$$

easy sol'n: $a = 1$

$b = -1$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

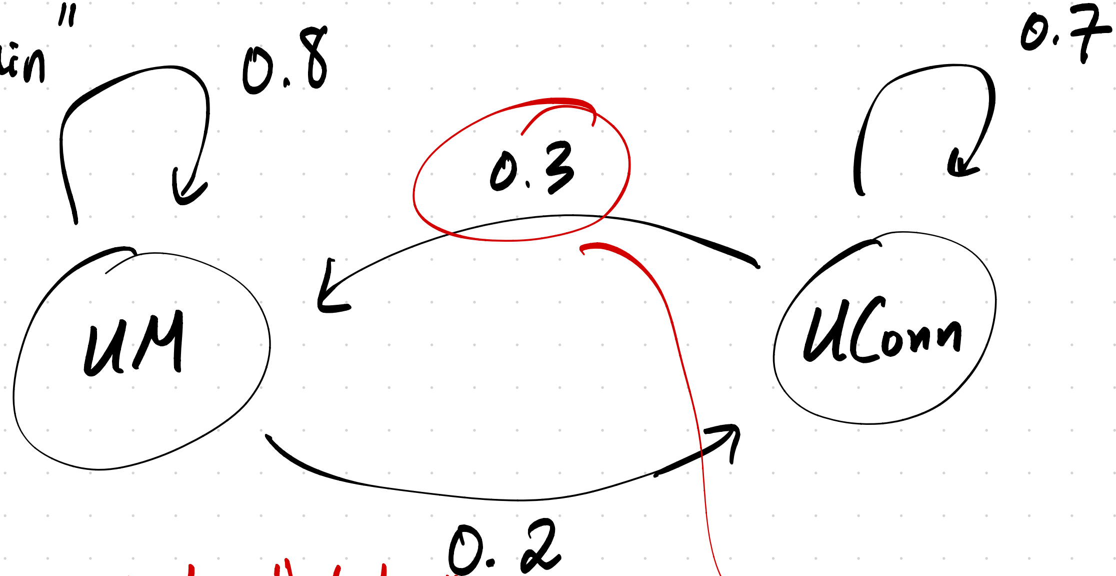
$$A\vec{v}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2\vec{v}_2 \checkmark$$

so, A has

$\lambda_1 = 5 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_2 = 2 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

"Markov chain"



nodes are called "states"

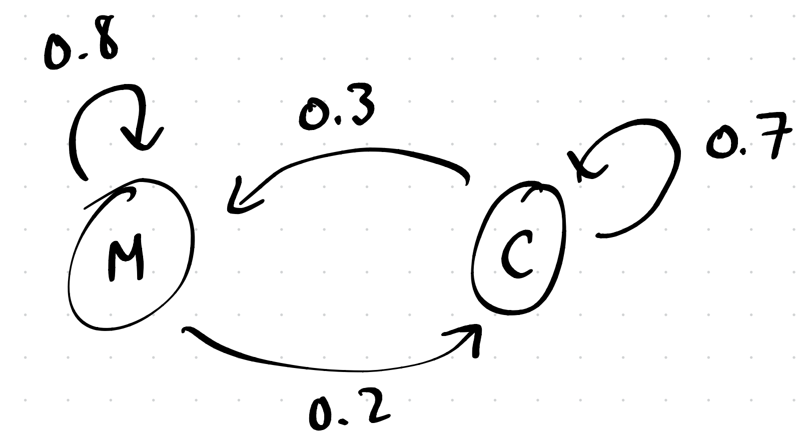
system is called a "Markov chain"

→ tomorrow only depends on today

probability that UM wins next year given that UConn wins this year

Q: what is the long-run fraction of time spent in each state?

i.e. what % of games will Michigan win, in the steady state?



adjacency matrix

$$A = \begin{array}{cc} \left[\begin{array}{cc} 0.8 & 0.3 \\ 0.2 & 0.7 \end{array}\right] & \begin{array}{l} \to UM \\ \to UConn \end{array} \\ \begin{array}{cc} UM\to & UConn\to \end{array} & \end{array}$$

columns add up to 1

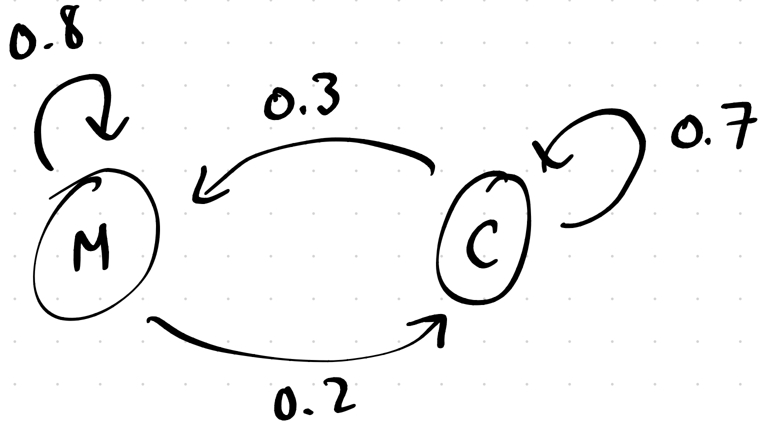
$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

"state vector" $\vec{x} \in \mathbb{R}^2$ $\quad \vec{x} = \begin{bmatrix} p(\text{Michigan}) \\ p(\text{UConn}) \end{bmatrix}$

let's start with an initial state vector,

$$\vec{x}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

initial state vector



$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

$$\vec{x}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

To advance the simulation by 1 step, multiply $\vec{x}_0$ by A!

$$\vec{x}_1 = A\vec{x}_0 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix}$$

To advance another step, multiply by A again!

$$\vec{x}_2 = A\vec{x}_1 = A\left(A\vec{x}_0\right)$$
$$= A^2\vec{x}_0$$

k steps into the future:

$$\vec{x}_k = A^k\vec{x}_0$$

as $k \to \infty$, what does $\vec{x}_k$ approach?

```python
def simulate_steps(A, x0, num_steps=15):
    x = x0
    for k in range(1, num_steps+1):
        # Note that np.linalg.matrix_power(A, k) is the same as A @ A @ ... @ A (
        # A ** k raises each element of A to the kth power, which is not what we
        x_k = np.linalg.matrix_power(A, k) @ x
        print(f'x_{k} = {x_k.flatten()}')

A = np.array([[0.8, 0.3],
              [0.2, 0.7]])

x0 = np.array([[1], [0]])

simulate_steps(A, x0)
```

```
x_1 = [0.8 0.2]
x_2 = [0.7 0.3]
x_3 = [0.65 0.35]
x_4 = [0.625 0.375]
x_5 = [0.6125 0.3875]
x_6 = [0.60625 0.39375]
x_7 = [0.603125 0.396875]
x_8 = [0.6015625 0.3984375]
x_9 = [0.60078125 0.39921875]
x_10 = [0.60039063 0.39960938]
x_11 = [0.60019531 0.39980469]
x_12 = [0.60009766 0.39990234]
x_13 = [0.60004883 0.39995117]
x_14 = [0.60002441 0.39997559]
x_15 = [0.60001221 0.39998779]
```

eigenvector of A, with $\lambda = 1$!

Big idea: If A is an adjacency matrix,

$$A^k \vec{x}_0 \to \text{an eigenvector for the eigenvalue } 1$$

conceptually, the long-run distribution vector $\vec{x}$ satisfies

$$A\vec{x} = \underset{1}{\uparrow}\vec{x}$$

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

Why do adjacency matrices <u>always</u> have an eigenvalue of 1?

- observe: cols add up to 1
- observe: $A^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- key: A and A^T have the same eigenvalues (same $p(\lambda)$ for both)

Important: It is true that all adjacency matrices have $\lambda = 1$ as an eigenvalue.

$\Rightarrow$ what's also true is $\lambda = 1$ is the largest eigenvalue for any adjacency matrix.

$\Rightarrow$ theorem linked in notes

$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ $\lambda_1 = 5$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

not an adjacency matrix! $\lambda_2 = 2$ $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Q: what does $A^k \vec{x}_0$ approach as $k \to \infty$?

A: The components of $A^k \vec{x}_0$ approach ∞, but the direction of $A^k \vec{x}_0$ approaches the direction of the dominant eigenvalue's eigenvector

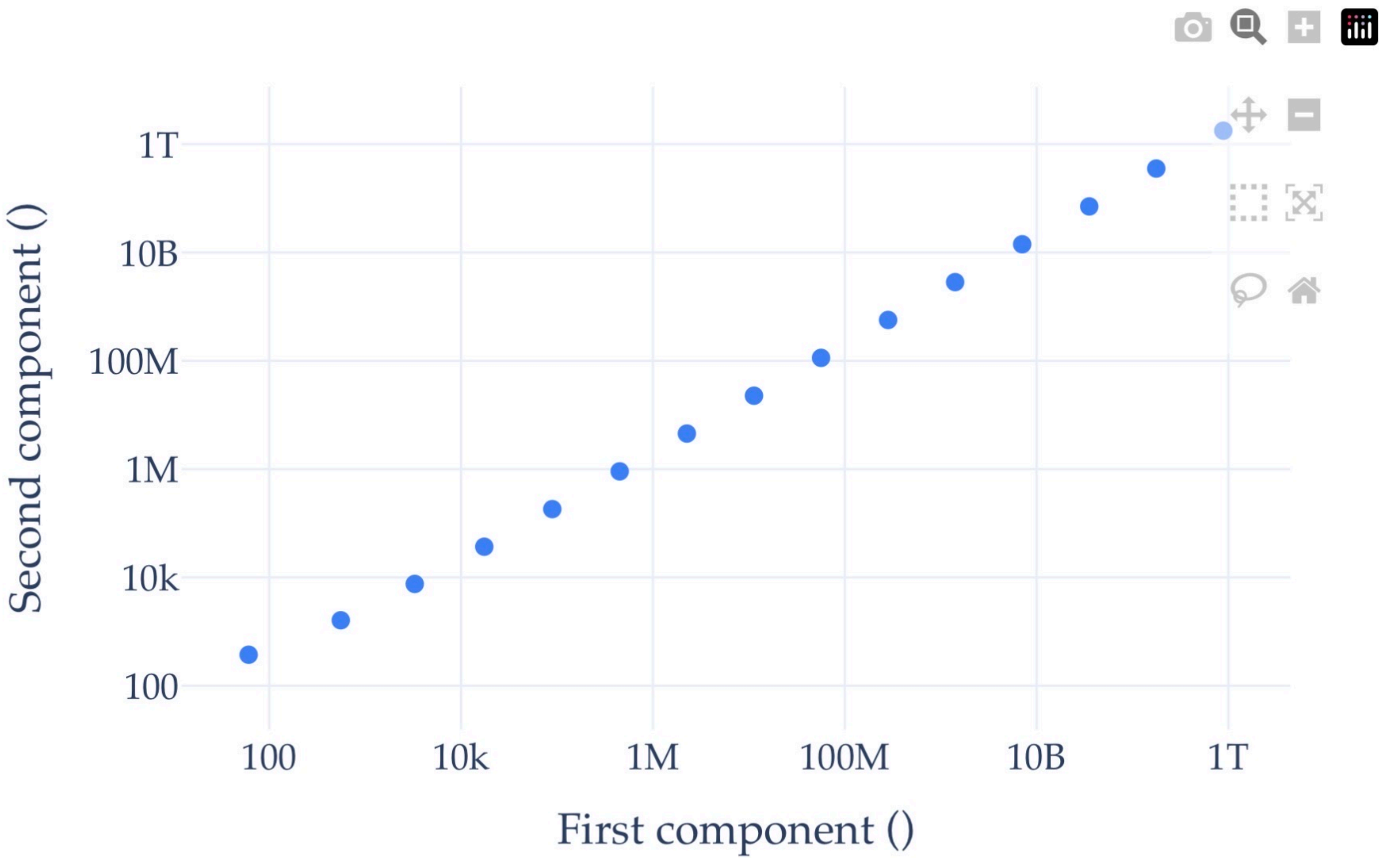
```
simulate_steps(
    A = np.array([[3, 1],
                  [2, 4]]),
    x0 = np.array([[-13], [100]])
)
```

```
x_1 = [ 61 374]
x_2 = [ 557 1618]
x_3 = [3289 7586]
x_4 = [17453 36922]
x_5 = [ 89281 182594]
x_6 = [450437 908938]
x_7 = [2260249 4536626]
x_8 = [11317373 22667002]
x_9 = [ 56619121 113302754]
x_10 = [283160117 566449258]
x_11 = [1415929609 2832117266]
x_12 = [ 7079906093 14160328282]
x_13 = [35400046561 70801125314]
x_14 = [177001264997 354004594378]
x_15 = [ 885008389369 1770020907506]
```

as $k \to \infty$,

$A^k \vec{x}_0 \to$ a scaled version of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A^{15} \begin{bmatrix} -13 \\ 100 \end{bmatrix}$$


1T
10B
100M
1M
10k
100
Second component ()
100
10k
1M
100M
10B
1T
First component ()

Why did $A^k \vec{x}_0$ converge in direction to the eigenvector with the largest eigenvalue?

Suppose $\vec{x} \in \mathbb{R}^2$

$\vec{v}_1, \vec{v}_2$ are linearly independent and span all of $\mathbb{R}^2$,

$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

not an adjacency matrix!

$\lambda_1 = 5 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_2 = 2 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

so it must be possible to write

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

Q: what happens when I multiply $\vec{x}$ by A?

$$A\vec{x} = A(c_1 \vec{v}_1 + c_2 \vec{v}_2)$$

$$= c_1 A\vec{v}_1 + c_2 A\vec{v}_2$$

$$= c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2$$

remember, $\vec{v}_1$, $\vec{v}_2$ are <u>eigenvectors</u> of A!

$$A^2 \vec{x} = c_1 \lambda_1^2 \vec{v}_1 + c_2 \lambda_2^2 \vec{v}_2$$

$$A^k \vec{x} = c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2$$

Q: As $k \to \infty$, what happens to $A^k \vec{x}$?

$$A^k \vec{x} = c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2$$

Just to illustrate, use $\lambda_1 = 5$, $\lambda_2 = 2$

$$\frac{A^k \vec{x}}{5^k} = \frac{c_1 5^k \vec{v}_1 + c_2 2^k \vec{v}_2}{5^k}$$

$$\frac{A^k \vec{x}}{5^k} = c_1 \vec{v}_1 + c_2 \left(\frac{2}{5}\right)^k \vec{v}_2$$

As $k \to \infty$, $\left(\frac{2}{5}\right)^k \to 0$

so direction of $A^k \vec{x} \to$ direction of $\vec{v}_1$!

In our analysis, we assumed that

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

$\vec{v}_1, \vec{v}_2$ were linearly independent and spanned $\mathbb{R}^2$

→ what if a matrix doesn't have n linearly independent eigenvectors?

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow$$ Find its eigenvalues and eigenvectors