EECS 245 Fall 2025

Math for ML

Lecture 20: Eigenvalues and Eigenvectors, Continued

Read Chapter 5.1!

Agenda

all about

eigen-things

vectors

values

spaces

bases

Announcements

→ MT 2 scores on Gradescope

→ 1 on 1 check-ins available

→ HW 10 coming Saturday

$$\underbrace{X^T X + \lambda I}_{\text{why is this always invertible?}}$$

square, $\underbrace{n \times n \text{ matrix } A}$

think about linear transformations from $\mathbb{R}^n \to \mathbb{R}^n$

(non-zero)

eigenvector $\vec{v}$:

$$A\vec{v} = \lambda \vec{v}$$

when multiplied by A, $\vec{v}$'s direction didn't change; it was just scaled by a factor of λ

eigenvalue

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

if $\vec{v}$ is an eigenvector of A, so is $c\vec{v}$, for any $c \neq 0$

Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A\vec{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ eigenvector of A with $\lambda_1 = 3$

$$A \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

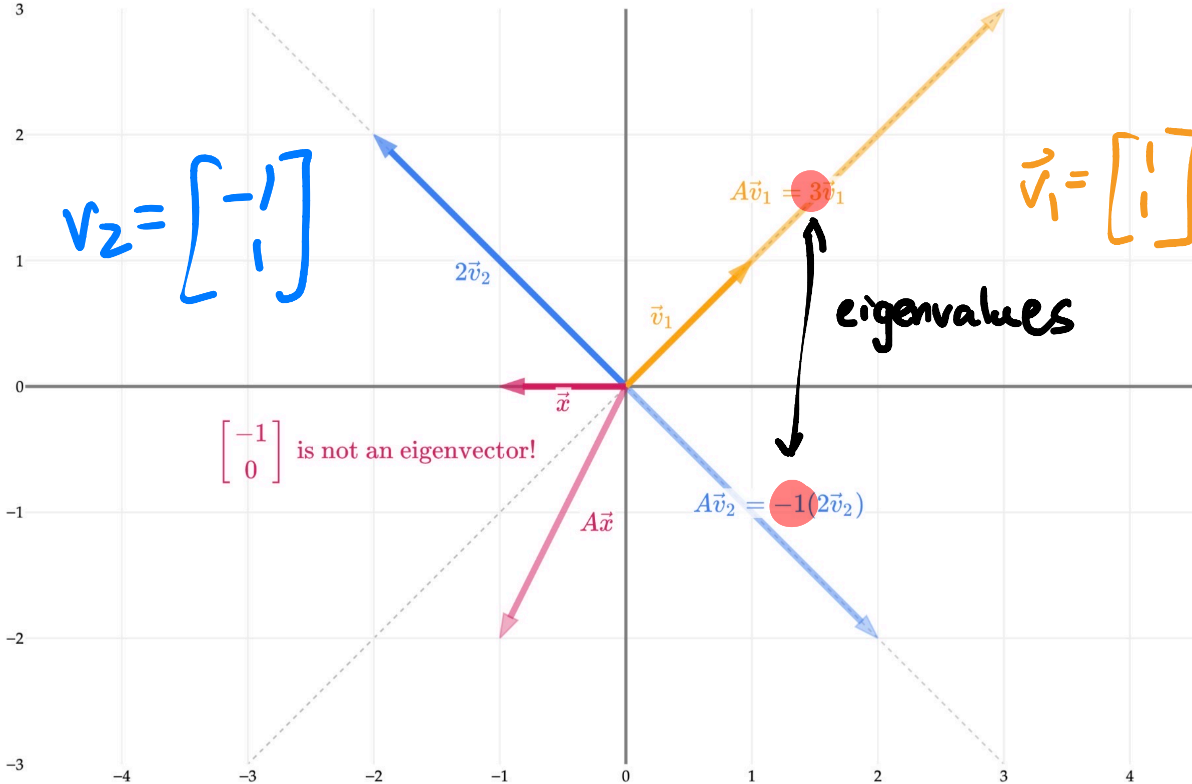
$\Rightarrow$ so $\vec{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is <u>also</u> eigvec of A with $\lambda = 3$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+2 \\ -2+1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

so, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = -1$

Visualizing the eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

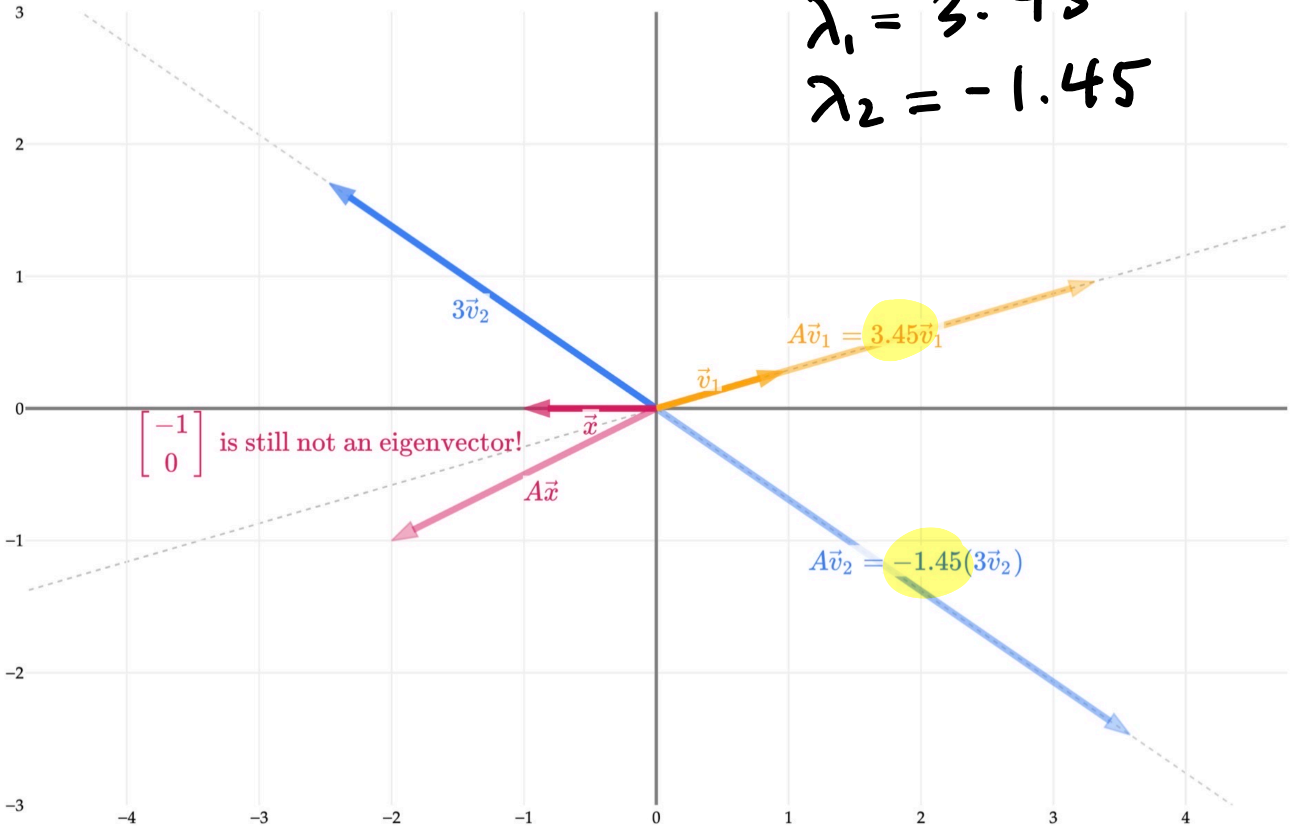


Visualizing the eigenvectors of $B = \begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix}$

Here,

$\lambda_1 = 3.45$

$\lambda_2 = -1.45$



```python
B = np.array([[2, 5],
              [1, 0]])

np.linalg.eig(B)
```

```
EigResult(eigenvalues=array([ 3.44948974, -1.44948974]), eigenvectors=array([[
0.96045535, -0.82311938],
       [ 0.27843404,  0.56786837]]))
```

$\vec{v}_1$ $\vec{v}_2$

cols are the eigvecs

observe: $\lambda_1 + \lambda_2 = 2$, which is sum of diagonal!

True in general: sum of λ_i's = sum of diagonal

$A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix}$

notice: $\text{rank}(A) = 1$

$\lambda_1 = 13$, $\lambda_2 = \underline{0}$?

$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

basis for nullsp(A)!

$\lambda_1 = 13$ ✓ $\begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13x \\ 13y \end{bmatrix}$

$x + 4y = 13x \Rightarrow 4y = 12x \Rightarrow y = 3x$

$3x + 12y = 13y \Rightarrow 3x = y$

both same, since any vector on that line works!

e.g. $x = 1$, so $y = 3$,

so $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

A has an eigenvalue of 0

$$\Updownarrow$$

A is <u>not</u> invertible

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ had

$\lambda_1 = 3$ with $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$ with $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

what are the eigenvalues of A^2?

what eigenvectors?

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

if $\vec{v}$ is an eigvec with eigval λ of A, then

$$A^2\vec{v} = AA\vec{v} = A(\lambda\vec{v}) = \lambda(A\vec{v})$$
$$= \lambda(\lambda\vec{v})$$
$$= \lambda^2\vec{v}$$

only one direction!

then, $\vec{v}$ is an eigvec of A^2 with eigval λ^2.

possible for A^2 to have an eigenvector that A doesn't have

think: rotations

characteristic polynomial

$$p(\lambda) = \det(A - \lambda I)$$

degree n polynomial

eigenvalues are solutions to

$$p(\lambda) = \det(A - \lambda I) = 0$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} \right)$$

$$= (1-\lambda)^2 - 4$$

$$= \lambda^2 - 2\lambda + 1 - 4$$

$$= \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3) = 0$$

$\lambda = -1$, $\lambda = 3$

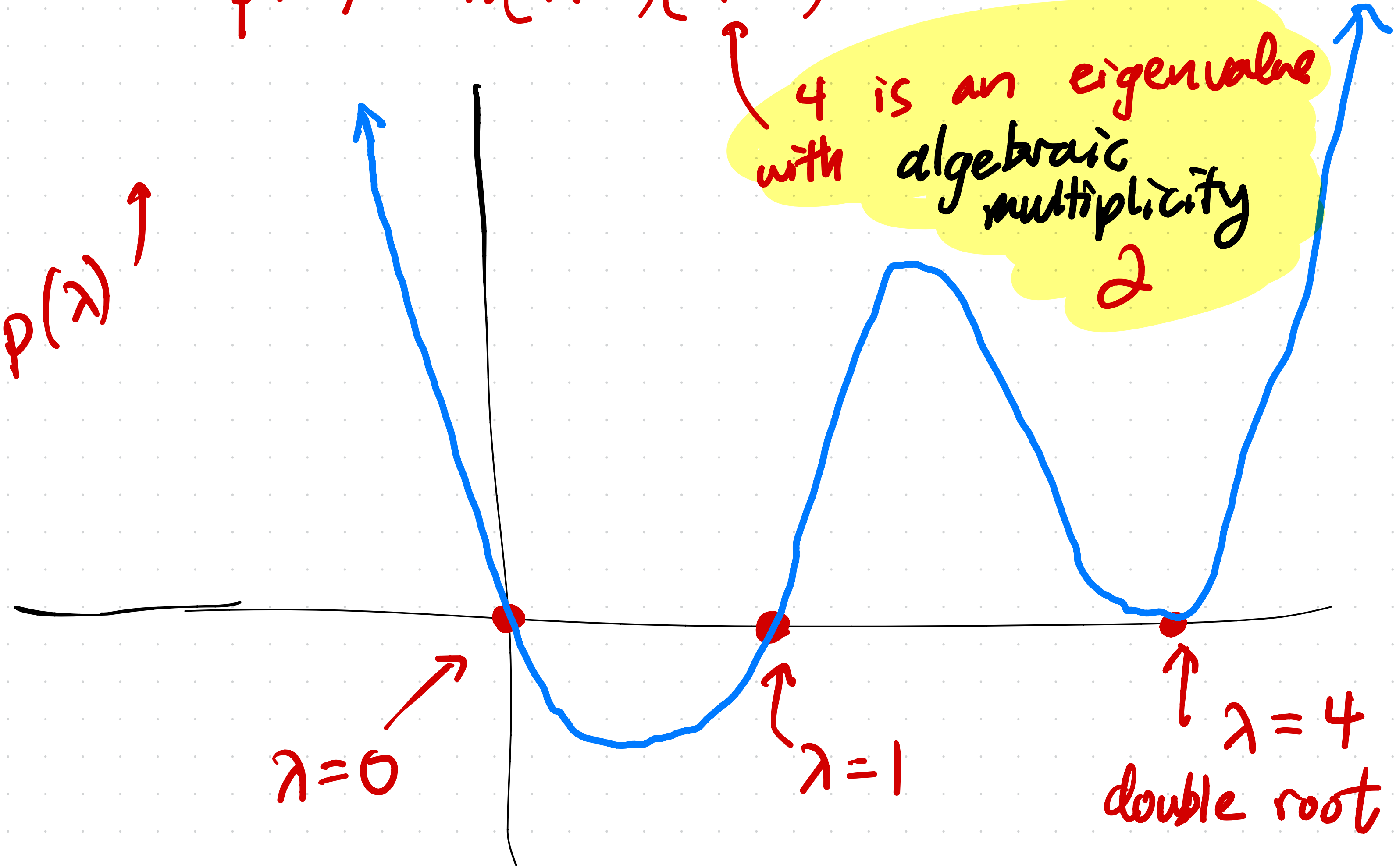
$$A = \begin{bmatrix} 4 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 4 \end{bmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} 4-\lambda & & & 0 \\ & 1-\lambda & & \\ & & 0-\lambda & \\ 0 & & & 4-\lambda \end{bmatrix}\right)$$

$$= (4-\lambda)(1-\lambda)(0-\lambda)(4-\lambda)$$

$$= \lambda(\lambda-1)(\lambda-4)^2$$

$$p(\lambda) = \lambda(\lambda-1)(\lambda-4)^2$$

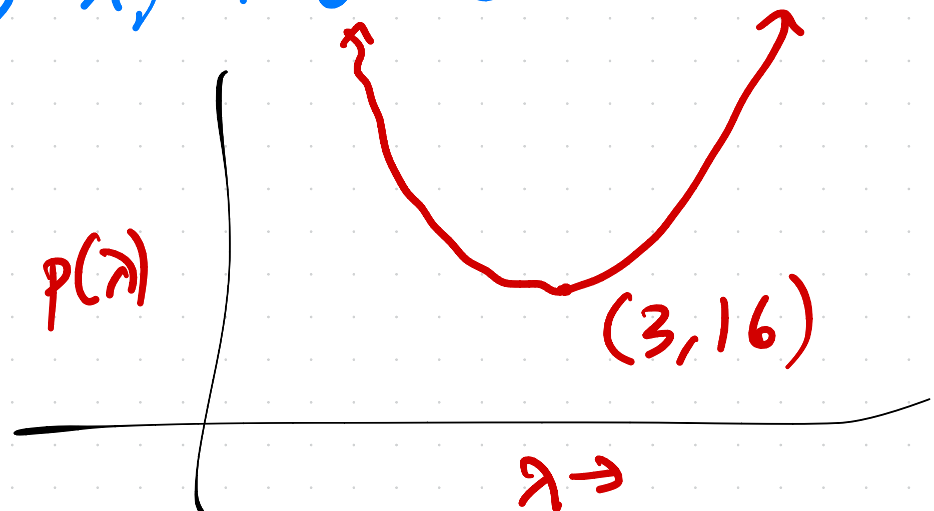


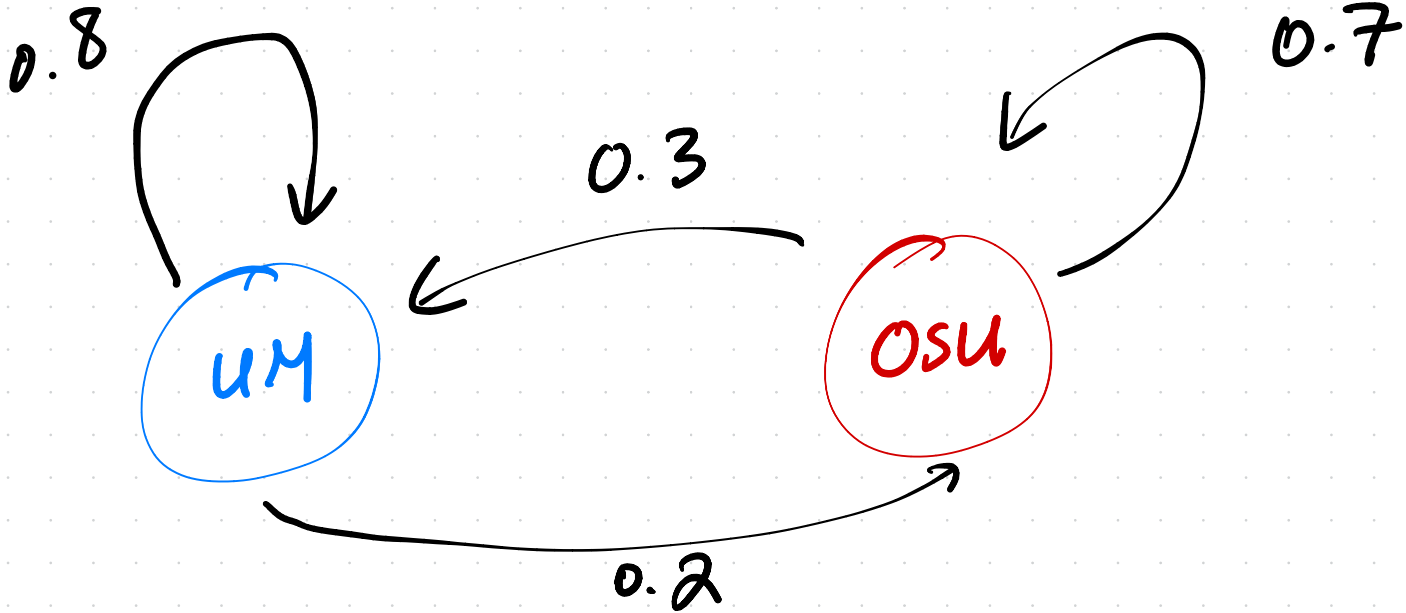
$$A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} = \underbrace{5 \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}}_{\text{rotation, then stretch}}$$

$$5e^{i\theta} \quad 5e^{-i\theta}$$

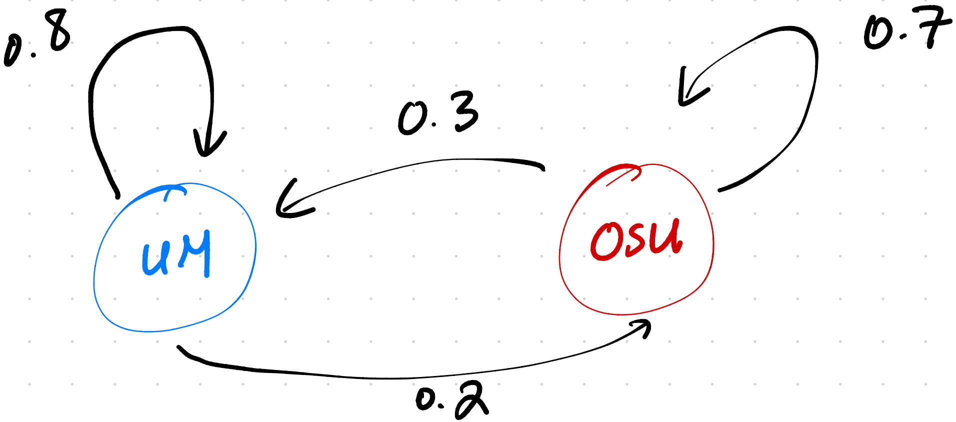
$$\theta = \cos^{-1}(3/5)$$

$$p(\lambda) = (3-\lambda)^2 + 16 = 0$$





in the long run, what % of games does Michigan win?



$$A = \begin{array}{cc} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} & \begin{array}{l} \to UM \\ \to OSU \end{array} \\ \begin{array}{cc} UM\to & \quad OSU\to \end{array} & \end{array}$$

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

$$\vec{x}_k = \begin{bmatrix} p(\text{Michigan})_k \\ p(\text{OSU})_k \end{bmatrix}$$

simulate!

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}_1 = A\vec{x}_0 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

$$\vec{x}_2 = A\vec{x}_1 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = A^2\vec{x}_0$$

in general,

$$\vec{x}_k = A^k \vec{x}_0$$

multiplying by A steps one iteration into the future!

--- big idea (see notes for code):

```
x_1 = [0.8 0.2]
x_2 = [0.7 0.3]
x_3 = [0.65 0.35]
x_4 = [0.625 0.375]
x_5 = [0.6125 0.3875]
x_6 = [0.60625 0.39375]
x_7 = [0.603125 0.396875]
x_8 = [0.6015625 0.3984375]
x_9 = [0.60078125 0.39921875]
x_10 = [0.60039063 0.39960938]
x_11 = [0.60019531 0.39980469]
x_12 = [0.60009766 0.39990234]
x_13 = [0.60004883 0.39995117]
x_14 = [0.60002441 0.39997559]
```

$\vec{x}_k \to$ an eigen vec of A

(corresponding to $\lambda = 1$)

$$A\vec{x} = 1\vec{x}$$

$$\vec{x} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left.\begin{aligned} 0.8x_1 + 0.3x_2 &= x_1 \\ 0.2x_1 + 0.7x_2 &= x_2 \end{aligned}\right\} \rightarrow \vec{x} = c\begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

add cond that

$$x_1 + x_2 = 1$$