EECS 245, Winter 2026

LEC 22

Diagonalization and the Spectral Theorem

→ Read Ch. 9.4 - 9.5

Agenda

Big idea: diagonalizable matrices have n linearly independent eigenvectors

- How do we know if a matrix is diagonalizable?
- Why do we care?

Announcements

- HW 10 out; due Tuesday
- MT 2 regrades due Thursday: come to office hours to chat!

```
simulate_steps(
    A = np.array([[3, 1],
                  [2, 4]]),
    x0 = np.array([[-13], [100]])
)
```

```
x_1 = [ 61 374]
x_2 = [ 557 1618]
x_3 = [3289 7586]
x_4 = [17453 36922]
x_5 = [ 89281 182594]
x_6 = [450437 908938]
x_7 = [2260249 4536626]
x_8 = [11317373 22667002]
x_9 = [ 56619121 113302754]
x_10 = [283160117 566449258]
x_11 = [1415929609 2832117266]
x_12 = [ 7079906093 14160328282]
x_13 = [35400046561 70801125314]
x_14 = [177001264997 354004594378]
x_15 = [ 885008389369 1770020907506]
```

$A\vec{x}$

$A^2\vec{x}$

$A^3\vec{x}$

$\vdots$

$A^{20}\vec{x}$

$\vdots$

$A^{100}\vec{x}$

<u>Last time</u>: we showed that if A is an $n \times n$ matrix and $\vec{x} \in \mathbb{R}^n$, then

as $k \to \infty$,

$A^k \vec{x} \to$ a scaled version of the eigenvector with the largest |eigenvalue|.

absolute value

Our analysis required us to assume that A's eigenvectors spanned all of $\mathbb{R}^n$, i.e. A has n linearly independent eigenvectors

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Find eigenvalues and eigenvectors

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$$

$\lambda = 1$ is a double root

$$\lambda_1 + \lambda_2 = 2$$

$$\lambda_1 \lambda_2 = 1$$

$$\Rightarrow \lambda_1 = 1, \ \lambda_2 = 1$$

$\Rightarrow$ repeated eigenvalue!

eigenvector must satisfy

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} a \\ b \end{bmatrix} = 1 \begin{bmatrix} a \\ b \end{bmatrix}$$

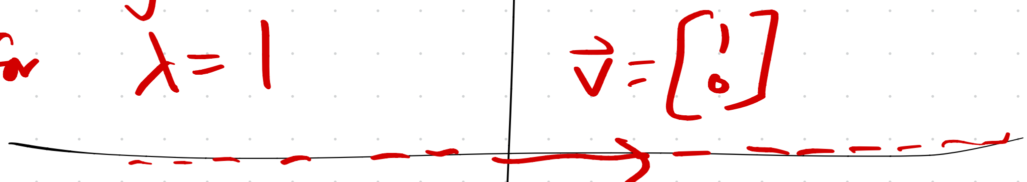
$$\Rightarrow \quad a + b = a, \quad b = b$$

$\Rightarrow$ $b = 0$, a free

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\mathbb{R}^2$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ only has one line of eigenvectors for $\lambda = 1$



$\Rightarrow$ A's eigenvectors don't span all of $\mathbb{R}^2$!

$\Rightarrow$ A is <u>not</u> diagonalizable!

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

also has $\lambda = 1$

repeated eigenvalue

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$a = a$$

$$b = b$$

both variables are free!

the set of all eigenvectors of A with $\lambda = 1$ is a 2-dimensional subspace of $\mathbb{R}^2$

(ignore the $\vec{0}$)

Eigenvalue decomposition of a matrix

Suppose A is an $n \times n$ matrix with n linearly independent eigenvectors

eigenvectors

$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$
$$A\vec{v}_2 = \lambda_2 \vec{v}_2$$
$$\vdots$$
$$A\vec{v}_n = \lambda_n \vec{v}_n$$

some λ_i's may be the same!

$$A \underbrace{\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}}_{V}$$

$$\underbrace{A\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}}_{V} = \begin{bmatrix} | & | & & | \\ A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$A V = V \Lambda$$

If A has n linearly independent eigenvectors,

then $V = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$

is invertible!

$$\Rightarrow \quad \boxed{A = V \Lambda V^{-1}}$$

eigenvalue decomposition of A

Application:

$$A = V \Lambda V^{-1}$$

Matrix powers are easy!

$$A^2 = V \Lambda \underbrace{V^{-1} V}_{I} \Lambda V^{-1}$$

$$= V \Lambda \Lambda V^{-1}$$

$$= V \Lambda^2 V^{-1}$$

$$\Lambda^2 = \begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{bmatrix}$$

$$A^{100} = V \Lambda^{100} V^{-1}$$

$$= V \begin{bmatrix} \lambda_1^{100} & & & \\ & \lambda_2^{100} & & \\ & & \ddots & \\ & & & \lambda_n^{100} \end{bmatrix} V^{-1}$$

way more efficient than multiplying

A by itself 100x !

Does every matrix have an eigenvalue decomposition

cols of V are eigvecs of A

$$A = V \Lambda V^{-1} \; ?$$

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

no: in order to be written in this form, A must have n linearly independent eigenvectors

$\Rightarrow$ A is diagonalizable if and only if it has n linearly independent eigenvectors

$\Rightarrow$ only diagonalizable matrices can be written $\boxed{A = V \Lambda V^{-1}}$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

not diagonalizable!

can't find 2 linearly independent eigenvectors

$$AM(1) = 2$$

Algebraic multiplicity of λ_i, denoted $AM(\lambda_i)$, the number of times λ_i appears as a root of $p(\lambda)$

degree n polynomial

$$p(\lambda) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}$$

$$AM(\lambda_i) = m_i$$

$$m_1 + m_2 + \cdots + m_k = n$$

Geometric multiplicity of eigenvalue λ_i, $GM(\lambda_i)$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{vs.} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$GM(1) = 2$

$GM(1) = 1$

For both of them, $AM(1) = 2$

but

$GM(1)$ is different

$GM(\lambda_i)$ = dimension of the subspace of all eigenvectors for eigenvalue λ_i

$$= \dim\left(\underbrace{\text{nullsp}(A - \lambda_i I)}_{\text{eigenspace for } \lambda_i}\right)$$

e.g. suppose 7 is an eigenvalue, $\vec{v}$ corresponding eigvec

$\vec{v}$: $A\vec{v} = 7\vec{v}$

$A\vec{v} - 7\vec{v} = \vec{0}$

$(A - 7I)\vec{v} = \vec{0}$

then $\vec{v} \in \text{nullsp}(A - 7I)$

For any λ_i,

$$AM(\lambda_i) \geq GM(\lambda_i) \geq 1$$

"potential" dimension of eigenspace

actual dimension

A diagonalizable iff $AM(\lambda_i) = GM(\lambda_i)$ for all λ_i's

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

"block" diagonal: eigvalues are eigvals of individual blocks

$(-1), (3), (3)$

$AM(-1) = 1$

$AM(3) = 2$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

upper triangular: eigvals on diag

eigvals: $2, 2, 3$

$AM(2) = 2$

$AM(3) = 1$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$(-1), (3), (3)$

$AM(-1) = 1$

$AM(3) = 2$

$p(\lambda) = (\lambda-3)^2(\lambda+1)$

eigenspace for $\lambda = 3$ is 2 dimensional!

$$= \text{nullsp}(A - 3I)$$

$$= \text{nullsp}\left(\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \text{span}\left(\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$$

$AM(3) = 2$

$GM(3) = 2$ because $\dim(\text{nullsp}(A - 3I)) = 2$

Therefore, A $\underline{\quad \text{is} \quad}$ diagonalizable!

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\underset{\sim}{B} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

upper triangular: eigvals on diag

eigvals: $2, 2, 3$

$$\dim(\operatorname{nullsp}(A - 2I))$$

$$= \dim\left(\operatorname{nullsp}\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\right)$$

$= 1$, because $\operatorname{rank}(A - 2I) = 2$ and $3 - 2 = 1$

$AM(2) = 2$

$AM(3) = 1$

$GM(2) = 1$

Therefore, B <u>is not</u> diagonalizable

Two thoughts

① Is diagonalizability related to invertibility?

No: A matrix could be either, both, or neither

② Symmetric matrices are a special case!

$$A = V\Lambda V^{-1} \Longrightarrow A = Q\Lambda Q^T$$

More in 9.5