EECS 245 Fall 2025
Math for ML

Lecture 22: Adjacency Matrices, Diagonalization

Read: Ch 5.1, 5.1 Part 2, 5.2 (all new)

Agenda

→ Recap: Adjacency matrices

  → Long-run behavior

→ The eigenvalue decomposition

→ Diagonalizability

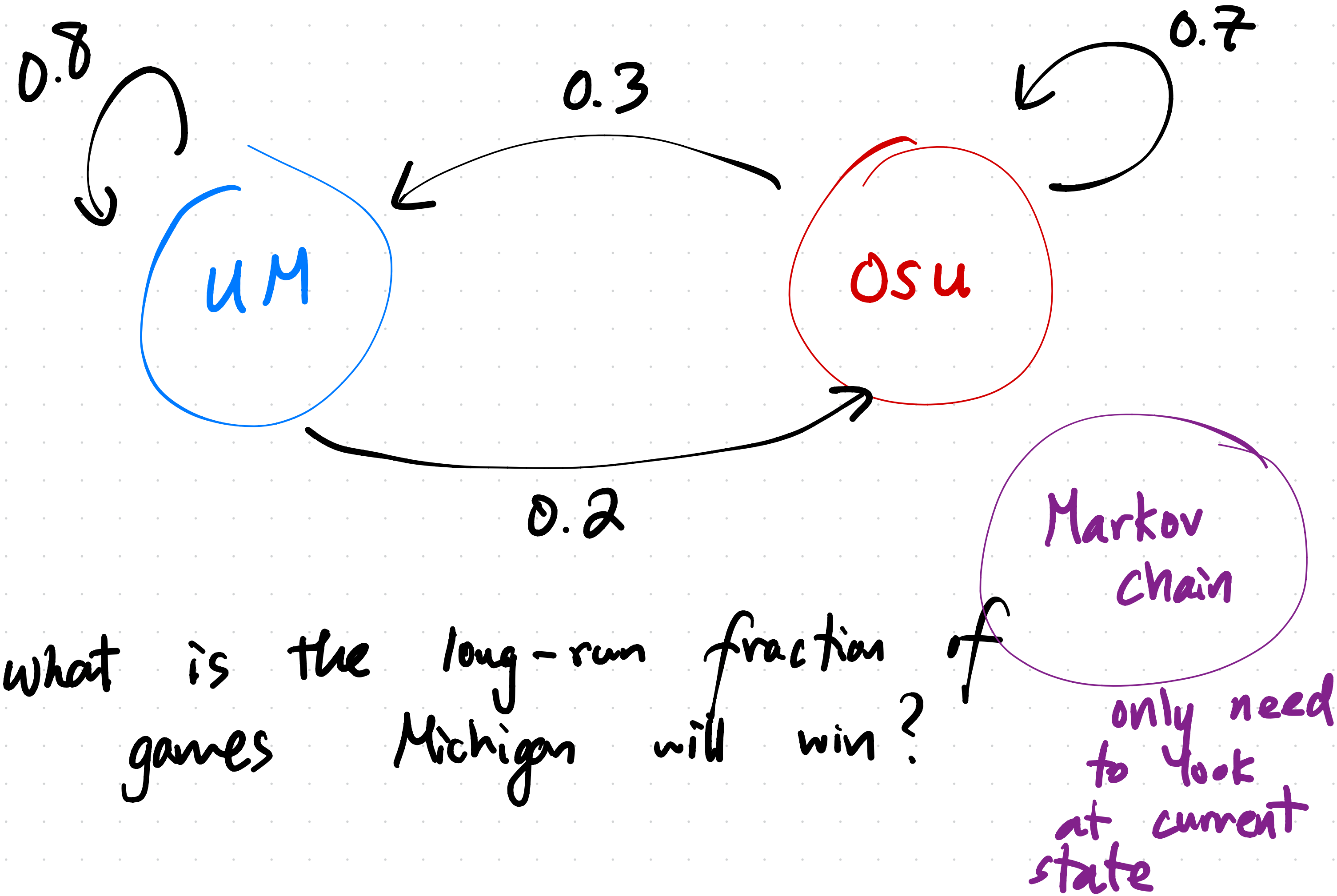
→ Multiplicities

Announcements

→ HW 10 due **Monday**

→ Midterm regrades due tomorrow

→ HW 9 grades coming soon



Q: what is the long-run fraction of games Michigan will win?

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

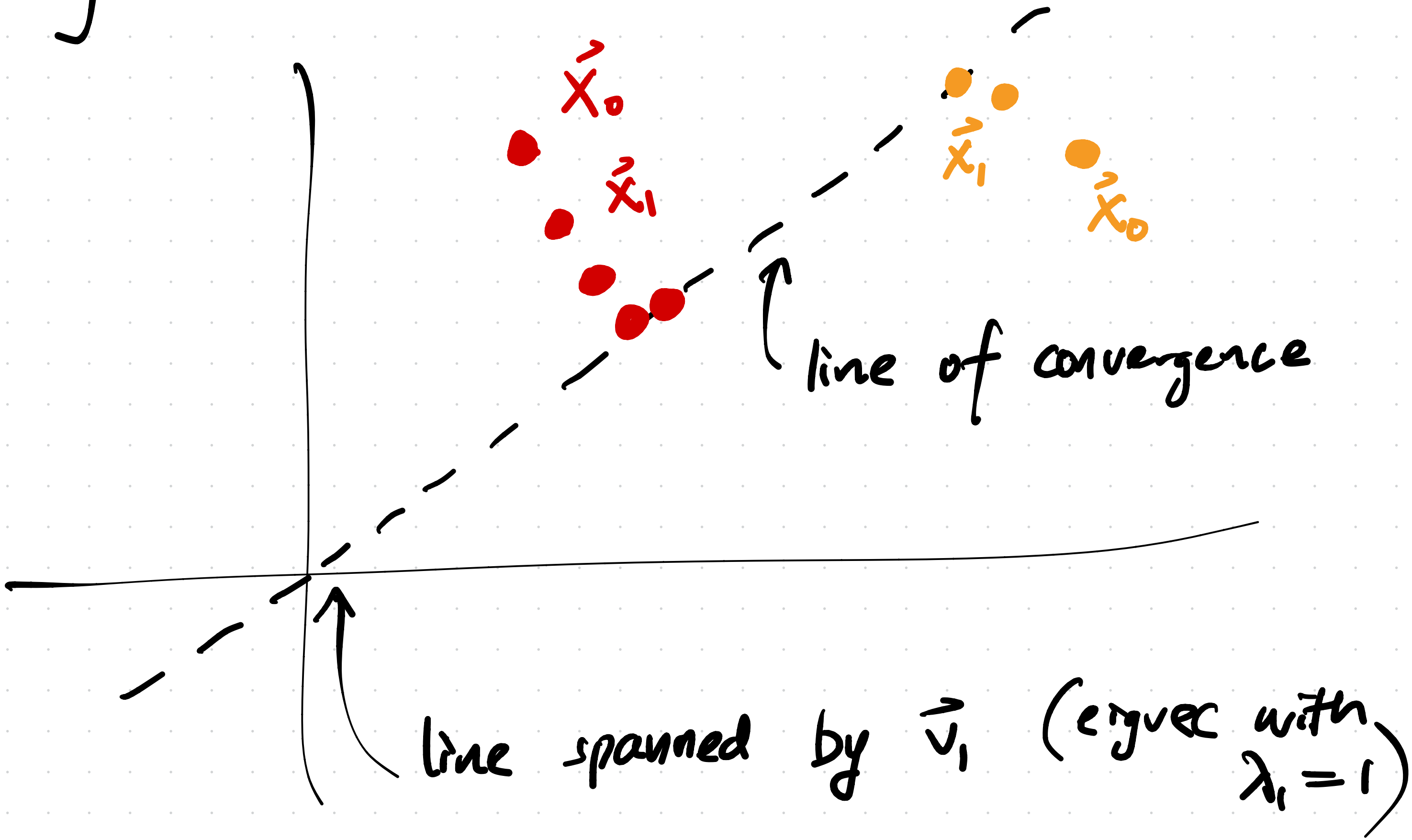
adjacency matrix: columns sum to 1, and all values $0 \le a_{ij} \le 1$

suppose we <u>simulate</u> $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ } "initial" state vec

what is $\underbrace{A\vec{x}_0}_{\text{"state vector"}}$? $A^2\vec{x}_0$? $A^3\vec{x}_0$? ....

... as $k \to \infty$, $A^k\vec{x}_0$?

big idea: $A^k \vec{x}_0 \longrightarrow$ the eigenvector with $\lambda = 1$



why?

span all of $\mathbb{R}^2$!

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

$$\lambda_1 = 1, \quad \vec{v}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$$\lambda_2 = 0.5, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0.5a \\ 0.5b \end{bmatrix}$$

$$A\vec{v} = 0.5\vec{v}$$

$$(A - 0.5I)\vec{v} = \vec{0}$$

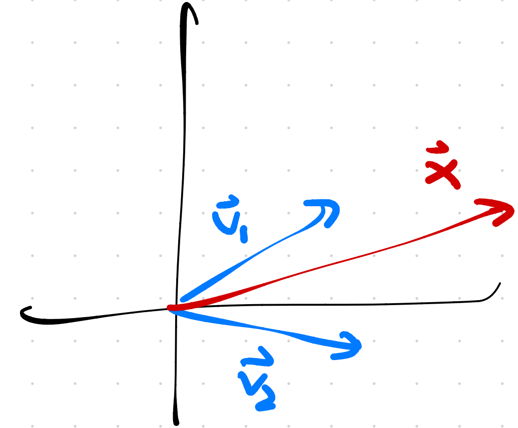
$$A - 0.5I = \begin{bmatrix} 0.8-0.5 & 0.3 \\ 0.2 & 0.7-0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{bmatrix} \to \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ in nullsp}$$

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

anything in $\mathbb{R}^2$

$$c_1 \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \end{bmatrix}$$



$$A\vec{x} = A\left(c_1 \vec{v}_1 + c_2 \vec{v}_2\right)$$

$$= c_1 A\vec{v}_1 + c_2 A\vec{v}_2$$

because $\vec{v}_1, \vec{v}_2$ are eigenvectors

$$= c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2$$

$$A^7 \vec{x} = c_1 \lambda_1^7 \vec{v}_1 + c_2 \lambda_2^7 \vec{v}_2$$

$$\vdots$$

$$A^k \vec{x} = c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2$$

$\lambda_1 = 1 \quad \vec{v}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$

$\lambda_2 = 0.5 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$A^k \vec{x} = c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2$$

$$= c_1 (1)^k \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + c_2 (0.5)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

as $k \to \infty$ $\left(0.5^k = \frac{1}{2^k}\right)$

$$\approx c_1 \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

no matter where we start, $A^k \vec{x}_0 \to \vec{v}_1$ (some multiple of)

Fact: Adjacency matrices always have a largest eigenvalue of 1.

1. Show that adjacency matrices always have an eigenvalue of 1

   details in notes
   IN SCOPE

2. Why is that always largest?

   "Perron-Frobenius theorem"
   not in scope

ch 2.9

$\vec{x} = A\vec{x}$

$\det(A^T) = \det(A)$

$\Rightarrow \det\left(\left(A - \lambda I\right)^T\right) = \det\left(A^T - \lambda I\right) = \det(A - \lambda I)$

$\Rightarrow$ A and A^T have the same eigvals!
(not necessarily eigvecs)

A^T has $\lambda = 1$, so does A!

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \qquad A^T = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

$$A^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8 + 0.2 \\ 0.3 + 0.7 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \qquad \lambda_1 = 2 \qquad \lambda_2 = 5$$

$$\vec{v}_1 \qquad \vec{v}_2$$

Let $\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$

$$A^k \vec{x} = c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2$$

$$\frac{A^k \vec{x}}{5^k} = \frac{c_1 2^k \vec{v}_1}{5^k} + \frac{c_2 5^k \vec{v}_2}{5^k}$$

$$\frac{A^k \vec{x}}{5^k} = c_1 \underbrace{\left(\frac{2}{5}\right)^k}_{\to 0 \text{ as } k \to \infty} \vec{v}_1 + c_2 \vec{v}_2$$

Takeaway: for <u>any</u> matrix A $n \times n$

with largest eigval λ_{max}

and corresponding eigvec $\vec{v}_{max}$,

$$A^k \vec{x} \longrightarrow \text{(some multiple of) } \vec{v}_{max}$$

any vector in $\mathbb{R}^n$

"power method"

ch 5.2

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \qquad \lambda_1 = 1 \quad \vec{v}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \qquad \lambda_2 = 0.5 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$AV = A\begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ A\vec{v}_1 & A\vec{v}_2 \\ | & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 \\ | & | \end{bmatrix} = V\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = V\Lambda$$

capital λ

$$AV = V\Lambda$$

This $V = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix}$ is invertible, so

2 LI eigvecs

$$\boxed{A = V\Lambda V^{-1}}$$

eigenvector decomposition

$$V = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

a matrix A is diagonalizable if there exists an invertible matrix P and diagonal matrix D such that

$$A = PDP^{-1}$$

how do we diagonalize? using

$$A = V \Lambda V^{-1}$$

if $A = V \Lambda V^{-1}$ doesn't exist, not diagonalizable ex

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad p(\lambda) = (1-\lambda)^2$$

only eigval is $\lambda = 1$

with algebraic multiplicity of 2

what eigvecs?

$$A \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$a + b = a \rightarrow b = 0$$

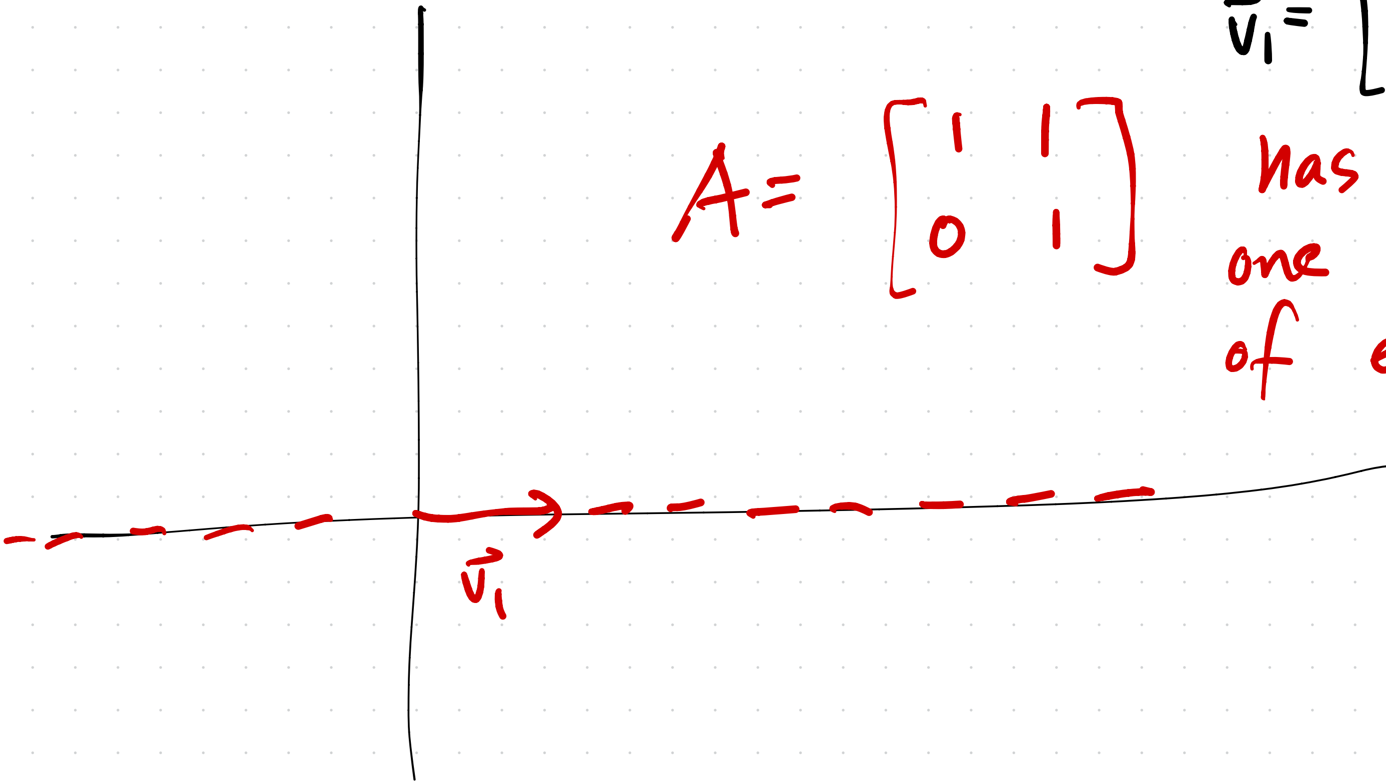
$$b = b$$

useless! ..

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

has only one line of eigvecs!



$$A = V \Lambda V^{-1}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 1 \\ \vec{v}_1 & \vec{v}_2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & ??? \\ 0 & ??? \end{bmatrix}$$

no second ind. eigvec exists!

A is not diagonalizable!

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$p(\lambda) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$

then the algebraic multiplicity of λ_i is m_i

$$m_1 + m_2 + \cdots + m_k = n$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad p(\lambda) = (1-\lambda)^2$$

algebraic mult is 2

$\dim(\text{nullsp}(A - \lambda_i I))$ = geom mult of λ_i

$$I \begin{bmatrix} 13 \\ -50 \end{bmatrix} = (1) \begin{bmatrix} 13 \\ -50 \end{bmatrix}$$

$V \Lambda V^{-1}$: $\quad V = \begin{bmatrix} 13 & 0 \\ -50 & 12 \end{bmatrix} \quad V \Lambda V^{-1} = I$