EECS 245, Winter 2026

LEC 23 Spectral Theorem, Singular Value Decomposition

→ Read Ch. 9.5, 10.1, 10.2

Agenda

- Spectral theorem } Ch. 9.5
  - → Last time, we discussed diagonalizable matrices,

$$A = V \Lambda V^{-1}$$

  - → Symmetric matrices are a special case!

$$A = Q \Lambda Q^{T}$$

- Singular value decomposition } Ch. 10
  - → How does this extend to non-square matrices?

Announcements

- HW 10 due tomorrow; HW 11 out tomorrow
- Final Exam in 2 weeks !!

Monday 4/27, 1:30-3:30PM

Activity 2

Suppose an $n \times n$ matrix A has the characteristic polynomial

$$p(\lambda) = (\lambda+1)^2 \underbrace{\lambda}_{(\lambda - 0)} (\lambda-1)^3(\lambda-4)^2(\lambda-5)(\lambda-12)^2$$

1. What is n (i.e. the number of rows/columns of A)? $n = 11$ (sum of algebraic mults.)
2. What is the determinant of A? 0, because 0 is an eigenvalue
3. What are all of A's eigenvalues and their algebraic multiplicities? $AM(-1) = 2$, for example
4. Is A diagonalizable?

Solution

we don't have enough information! need to know the geometric multiplicities of each eigenvalue

A diagonalizable matrix A can be decomposed:

$$A = V \Lambda V^{-1}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{bmatrix}$$

cols of V are eigenvectors of A

symmetric matrices are special! $A = A^T$

suppose A is an $n \times n$ symmetric matrix with real-valued entries. Then:

① A has n real-valued eigenvalues (factoring in algebraic multiplicity)

② Eigenvectors for different eigenvalues are orthogonal! → see Lab 12 or Ch. 9.5 for proof

③ For all λ_i, $AM(\lambda_i) = GM(\lambda_i)$

① + ② + ③ : Any symmetric matrix A can be "diagonalized" by an orthogonal matrix, Q

$$A = V \Lambda V^{-1} \Rightarrow A = Q \Lambda Q^T$$

$$Q^{-1} = Q^T$$

$$Q = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

These $\vec{v}_i$'s, if they correspond to different λ_i's, are automatically orthogonal!

If they correspond to the same λ_i, just pick them to be orthogonal

$\Rightarrow$ Make each $\vec{v}_i$ a unit vector $\Rightarrow$ $\boxed{Q^TQ = QQ^T = I}$

$$A = Q \Lambda Q^T$$

better than $A = V \Lambda V^{-1}$

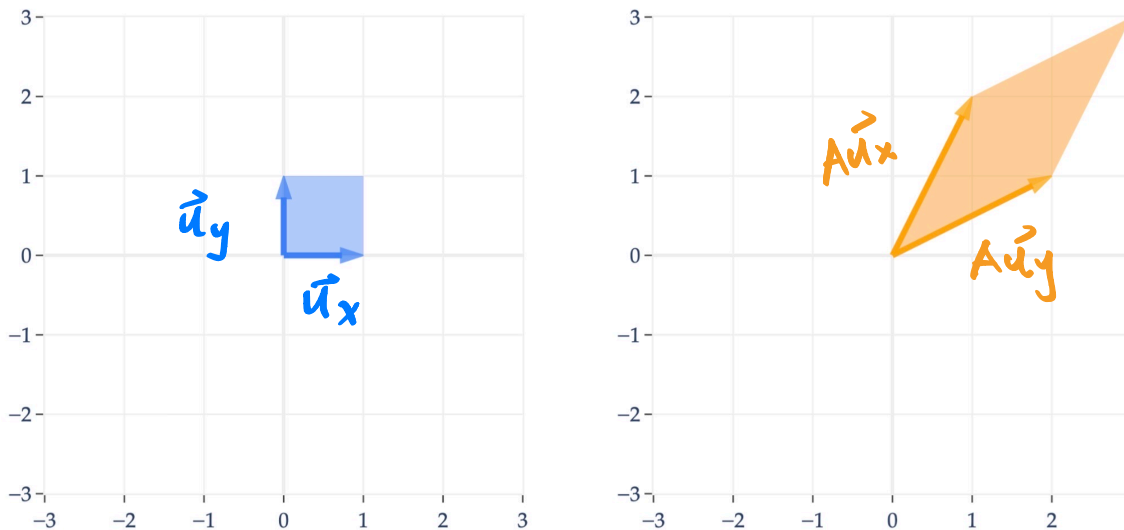
because Q^T is easy to find!

V^{-1} harder to find

$$f(\vec{x}) = A\vec{x} = Q\Lambda Q^T\vec{x}$$

Let's make sense of this visually. Consider the symmetric matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Transformation by A

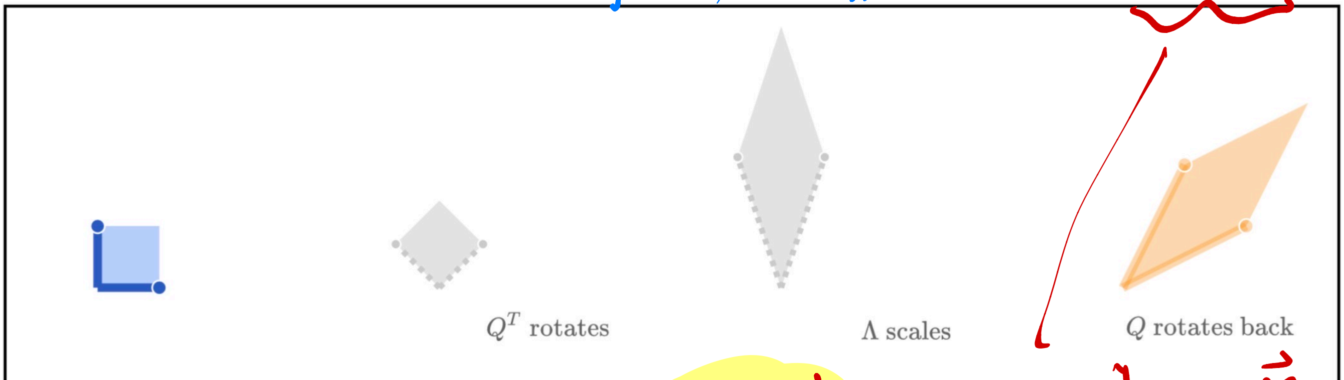


$Q\begin{bmatrix}3\\-5\end{bmatrix}$ vs. $Q^T\begin{bmatrix}3\\-5\end{bmatrix}$

$$A = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} = Q\Lambda Q^T$$

$$f(\vec{x}) = A\vec{x} = Q\Lambda \underbrace{Q^T\vec{x}}$$

Visualizing $A = Q\Lambda Q^T$



remember, $Q^T = Q^{-1}$

writes $\vec{x}$ as a linear combination of the eigenvectors!

$$Q\begin{bmatrix}3\\-5\end{bmatrix} = \text{linear combination of } A\text{'s eigenvectors}$$

$$= 3\vec{v}_1 - 5\vec{v}_2$$

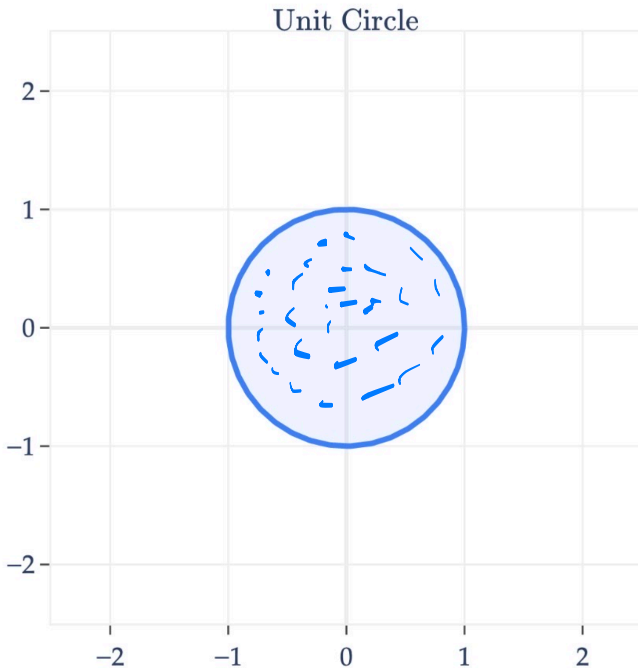
eigvecs of A

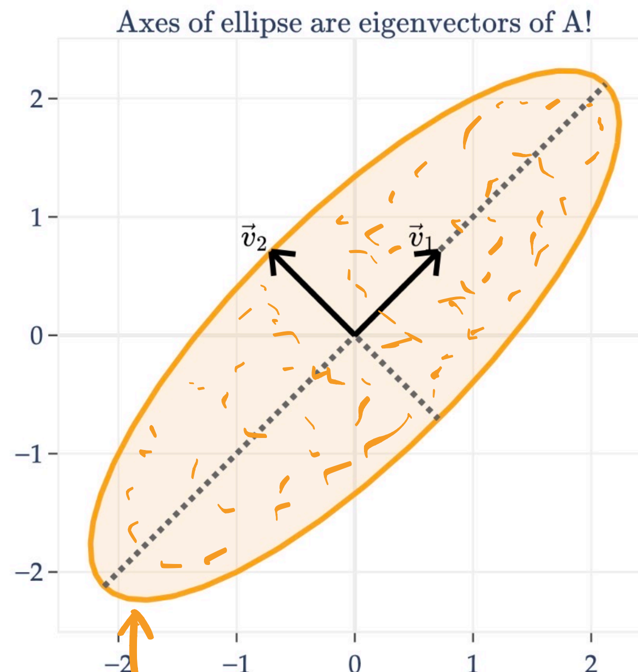
$$Q^T\begin{bmatrix}3\\-5\end{bmatrix} = Q^{-1}\begin{bmatrix}3\\-5\end{bmatrix} =$$ how do I make $\begin{bmatrix}3\\-5\end{bmatrix}$ out of $\vec{v}_1, \vec{v}_2$?

$$= \begin{bmatrix}a\\b\end{bmatrix}$$

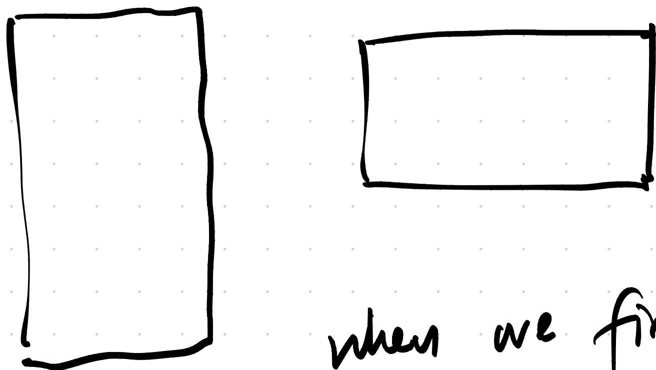
Then $a\vec{v}_1 + b\vec{v}_2 = \begin{bmatrix}3\\-5\end{bmatrix}$

$$f(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x}$$





eigvals are 3 and −1



what about non-square matrices?

when we first introduced eigenvalues and eigenvectors,

A : $n \times n$ matrix

$$A\underbrace{\vec{v}}_{n\times 1} = \underbrace{\lambda \vec{v}}_{n\times 1}$$

Suppose X is an $n \times d$ matrix

right singular vector, because $X\vec{v}$: $\vec{v}$ is on the right of X

left singular vector

$$\underset{n\times d}{X}\ \underset{d\times 1}{\vec{v}_i} = \sigma_i\ \underset{n\times 1}{\vec{u}_i}$$

"singular value"

$\vec{v} \in \mathbb{R}^d$, $\vec{u} \in \mathbb{R}^n$

Singular value decomposition

exists for every matrix $X : n \times d$

important: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ order from big to small

$$X_{n \times d} = U_{n \times n} \Sigma_{n \times d} V^T_{d \times d}$$

U orthogonal: $U^T U = U U^T = I_{n \times n}$

V orthogonal: $V^T V = V V^T = I_{d \times d}$

columns of U are the u_i's, called left singular vectors

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 & 0 \\ & \sigma_2 & & 0 \\ & & \ddots & 0 \\ 0 & & \sigma_r & 0 \\ & & & 0 \end{bmatrix}$$

singular values on diagonal

columns of V are the v_i's, called right singular vectors

$$X = \begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 5 \\ 2 & -2 & 0 \\ 5 & 5 & 10 \end{bmatrix}$$

observe:

rank(X) = # non-zero σ_i's $= 2$

$$X = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{2}{3} \\ \frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}}_{V^T}$$

$$X \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = 15 \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \end{bmatrix}$$

$$X\vec{v}_1 = 15\,\vec{u}_1$$

cols of V = rows of V^T

Where did the SVD of X come from?

$$X_{n \times d} = U \Sigma V^T$$

X not square, but ..... $X^T X$ and $X X^T$ both are!

→ bonus : $X^T X$ and $X X^T$ are also symmetric!!!!

$$X = U\Sigma V^T$$

$$X^TX = (U\Sigma V^T)^T U\Sigma V^T$$

$$= V\Sigma^T \underbrace{U^T U}_{I} \Sigma V^T$$

$$\boxed{\begin{aligned} X^TX &= V\Sigma^T\Sigma V^T \\ XX^T &= U\Sigma\Sigma^T U^T \end{aligned}}$$

these are just $Q\Lambda Q^T$ for X^TX and XX^T!!!

Takeaway: columns of U are eigenvectors of XX^T

columns of V are eigenvectors of X^TX

each non-zero $\sigma_i = \sqrt{\lambda_i}$, where

λ_i = eigenvalue of X^TX

= eigenvalue of XX^T

Computing the SVD By Hand

To conclude, we found $X = U\Sigma V^T$ by:

1. Computing X^TX.
2. Finding the eigenvalues of X^TX; their square roots are the singular values of X
   .

   $$\sigma_i = \sqrt{\lambda_i}$$

   These singular values are placed in the diagonal of Σ in decreasing order.
3. For each eigenvalue λ_i, finding an orthonormal eigenvector $\vec{v}_i$ of X^TX and placing it in the ith column of V.
4. For each $i = 1, 2, ..., r$, finding $\vec{u}_i$ by solving

   $$\vec{u}_i = \frac{1}{\sigma_i} X\vec{v}_i$$

   and placing it in the ith column of U.
5. Filling the rest of U with orthonormal vectors that form a basis for $\text{nullsp}(X^T)$.

This is not the only possible sequence of steps to follow; for instance, once you find the singular values $\sigma_1, \sigma_2, ..., \sigma_r$, you can independently find orthonormal eigenvectors of X^TX and XX^T and use them to form U and V. Just make sure you place the $\vec{u}_i$'s and $\vec{v}_i$'s in the correct order, corresponding to the order of the singular values in Σ.