EECS 245 Fall 2025
Math for ML

Lecture 23: Diagonalization, Spectral Theorem, SVD

Read: Ch 5.2, Lab 11 solutions

Agenda

→ Recap: diagonalization, multiplicities

→ The spectral theorem

→ Towards the singular value decomposition

the GOAT decomposition!

Announcements

→ HW 10 due Monday

→ Read Lab 11 solutions!

Extension of Ch. 5.2.

→ HW 9 scores up

Activity 1

$\lambda = 0$ is an eigval!

Suppose an $n \times n$ matrix A has the characteristic polynomial

$$p(\lambda) = (\lambda+1)^2 \lambda (\lambda-1)^3 (\lambda-4)^2 (\lambda-5)(\lambda-12)^2$$

1. What is n (i.e. the number of rows/columns of A)? 11 : sum of exponents
2. What is the determinant of A? $\det(A) = 0$; $\det(A) =$ product of λ's
3. What are all of A's eigenvalues and their algebraic multiplicities?

Solution

$\lambda = -1$, $AM(-1) = 2$

$\lambda = 0$, $AM(0) = 1$

$\vdots$

find from exponents

diagonalizable? not enough info!

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$GM(\lambda_i) = \dim\Big(\underbrace{\text{nullsp}(A - \lambda_i I)}_{\text{"eigenspace" for } \lambda_i:\ \text{set of all eigenvectors for } \lambda_i}\Big)$$

$\lambda_1 = 3 \quad AM(3) = 2$

$\lambda_2 = -1 \quad AM(-1) = 1$

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)\big((1-\lambda)(3-\lambda)\big) - 2\big(2(3-\lambda)\big)$$

$$= (3-\lambda)\left[(1-\lambda)^2 - 4\right]$$

$$= (3-\lambda)(\lambda^2 - 2\lambda - 3)$$

$$= (3-\lambda)(\lambda+1)(\lambda-3)$$

$\underline{\lambda_1 = 3}$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rank 1!

$$\vec{v} = \begin{bmatrix} 7 \\ 7 \\ -11 \end{bmatrix}$$

$$A\vec{v} = 3\vec{v}$$

$$\dim(\text{nullsp}(A - 3I)) = 2 = GM(3)$$ ✓

$$\text{nullsp}(A - 3I) = \text{span}\left(\left\{ \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}\right)$$

eigenspace for $\lambda = 3$

$\lambda_2 = -1$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\text{nullsp}(A+I) = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\dim(\text{nullsp}(A+I)) = 1 = GM(-1)$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\text{nullsp}(A+I) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\}\right)$$

For $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\lambda_1 = 3 \quad \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda_2 = -1 \quad \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

A has 3 linearly independent eigenvectors! $\Longleftrightarrow$ A diagonalizable!

$\Longleftrightarrow$ $AM(\lambda_i) = GM(\lambda_i)$ for all λ_i

$$A = V \Lambda V^{-1}$$

eigenvector matrix

eigval

$$V = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

matrix powers are easy!

$$A = V \Lambda V^{-1}$$

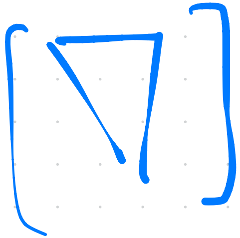
$$A^{10} = V \Lambda \underbrace{V^{-1} V}_{I} \Lambda \underbrace{V^{-1} V}_{I} \Lambda V^{-1} \cdots V \Lambda V^{-1}$$

$$= V \Lambda^{10} V^{-1} \qquad \Lambda^{10} = \begin{bmatrix} 3^{10} & & \\ & 3^{10} & \\ 0 & & -1^{10} \end{bmatrix}$$

diagonal: easy to exponentiate!

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

slightly different from last example!



so λ's are on diagonal:

$$\lambda_1 = 2 \quad AM(2) = 2$$

$$\lambda_2 = 3 \quad AM(3) = 1$$

$\underline{\lambda_1 = 2}$

$$A - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

AM > GM

can't find 3 LI eigvecs

$\dim(\text{nullsp}(A - 2I)) = 1$ which is less than $AM(2) = 2$ $\longrightarrow$ not diagonalizable!

"Spectral Theorem" suppose A is symmetric

then all of these are true: $A = A^T$

① A has exactly n real eigvals (none are complex)

② the eigvecs for different eigvals are orthogonal

③ for all λ_i, $AM(\lambda_i) = GM(\lambda_i)$

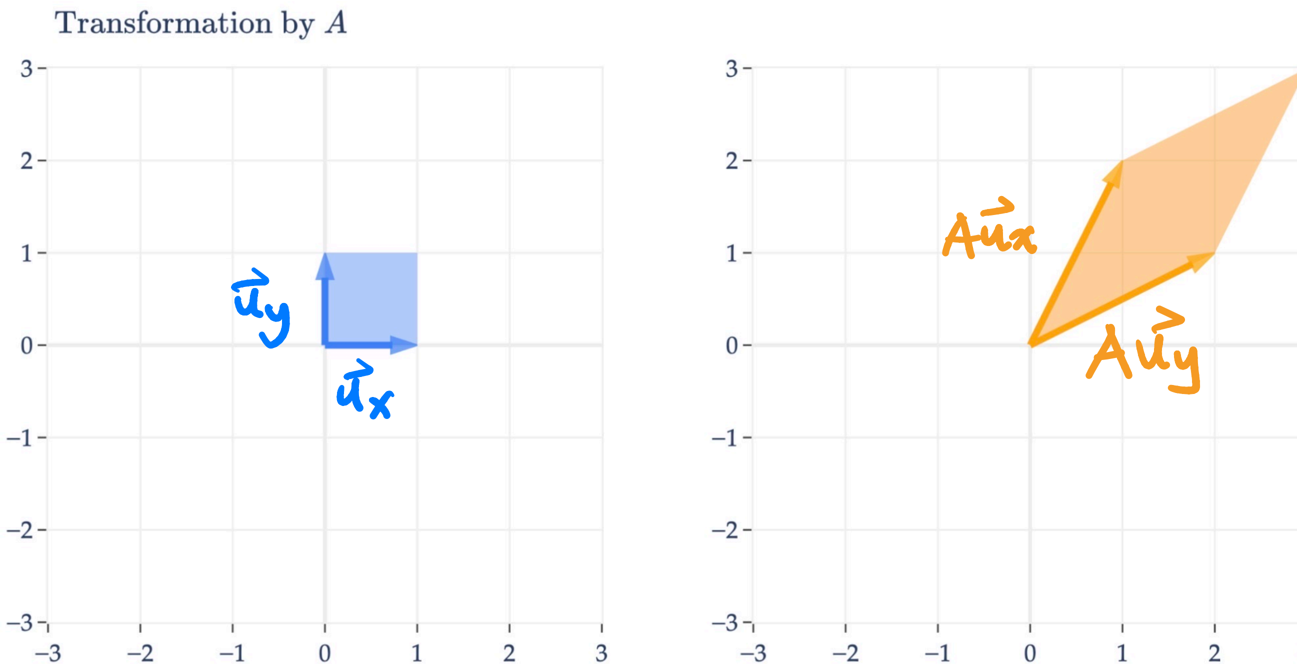
①, ②, ③ $\Rightarrow$ $A = V \Lambda V^{-1}$

there exists Q such that $\Rightarrow$ $A = Q \Lambda Q^T$

Q orthogonal

$Q^T Q = Q Q^T = I$

Let's make sense of this visually. Consider the symmetric matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.



$$A = Q\Lambda Q^T$$

$$f(\vec{x}) = A\vec{x} = Q\Lambda Q^T\vec{x}$$

$\mathbb{R}^n \to \mathbb{R}^n$

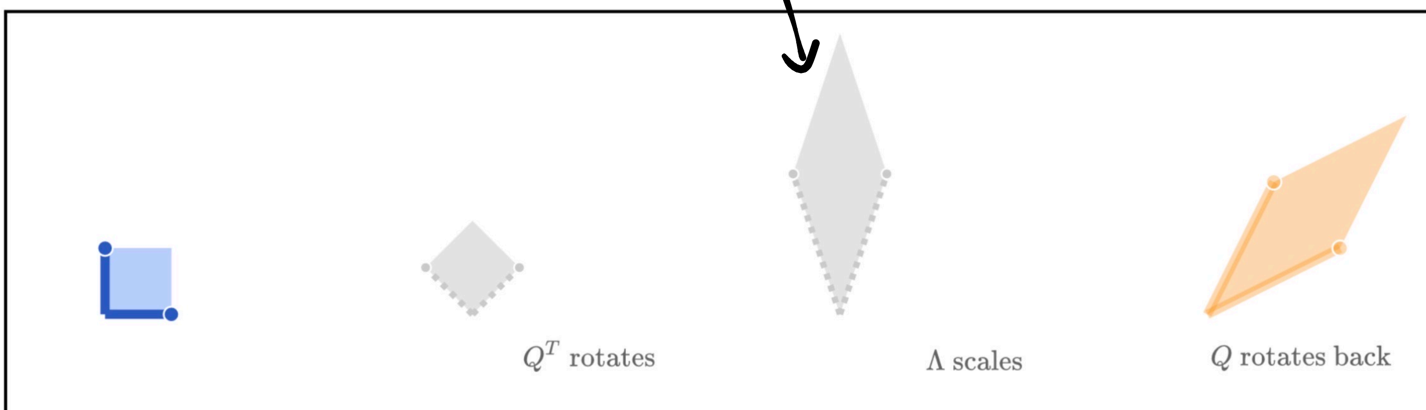
Q : rotates

Q^T : opposite rotation

Λ : scales/stretches

stretched 3x vertically, bc one $\lambda_1 = 3$

Visualizing $A = Q\Lambda Q^T$



Remember, $Q^T = Q^{-1}$!

Difference between Q and Q^T

$$Q = \begin{bmatrix} | & | \\ \vec{q}_1 & \vec{q}_2 \\ | & | \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

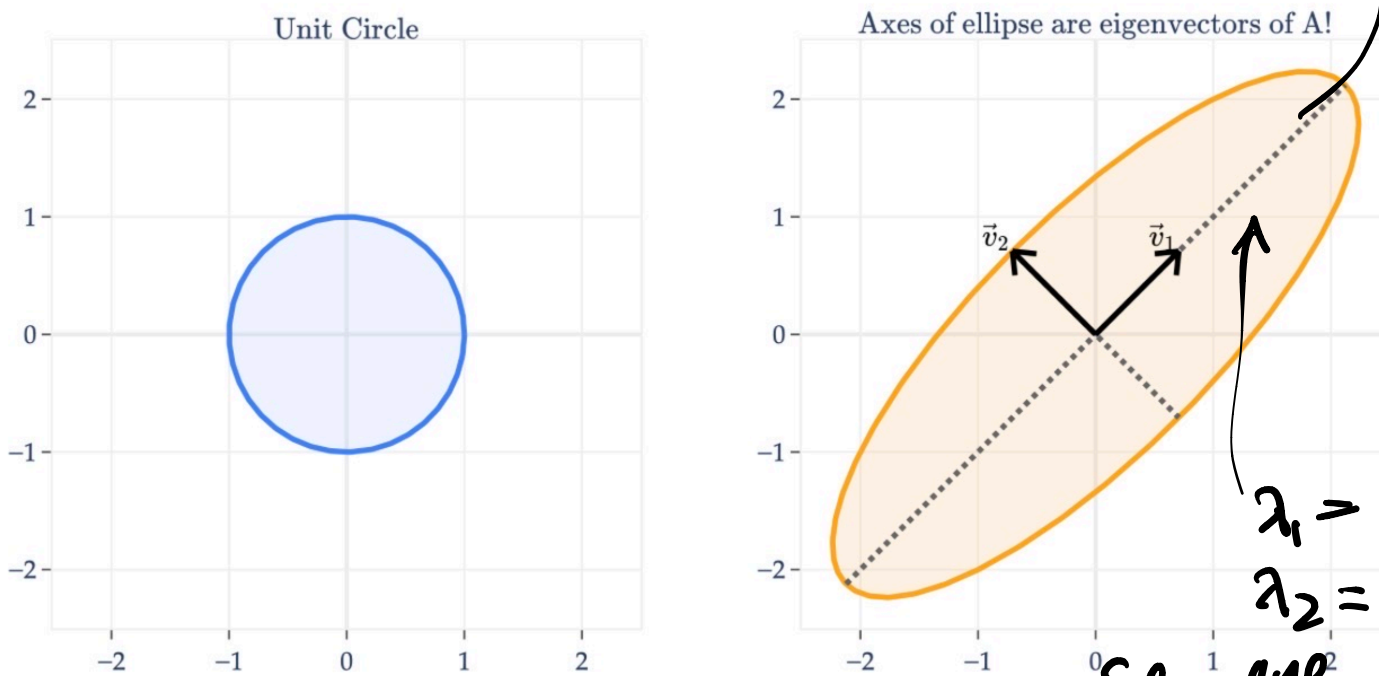
$$Q\vec{x} = 2\vec{q}_1 + 3\vec{q}_2 + \dots$$ = lin comb of the eigvecs

$$Q^T\vec{y} = \vec{z}$$
$$\vec{y} = Q\vec{z}$$

$\Rightarrow$ $Q^T\vec{y}$ tells you how much of each eigvec to mix to make $\vec{y}$!

axes of ellipse are determined by eigvectors!

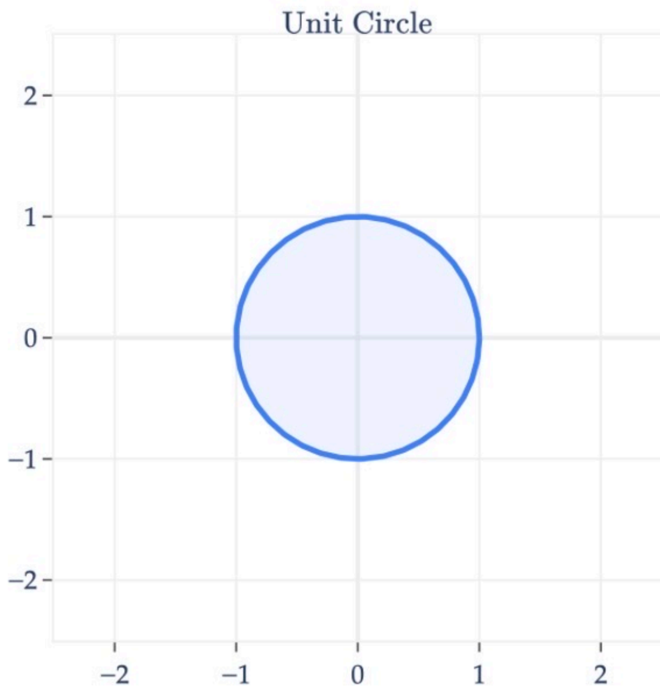
$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$



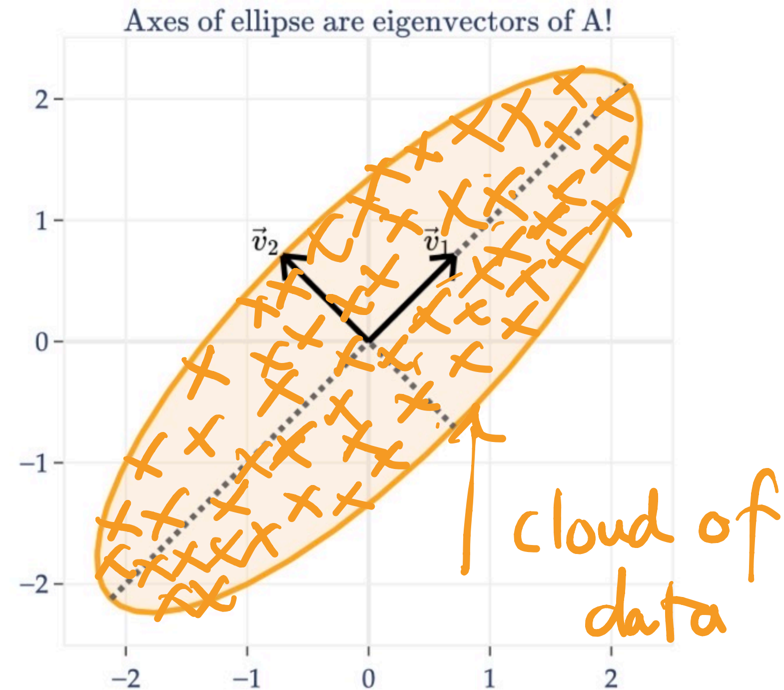
$\lambda_1 = 3$,
$\lambda_2 = -1$,
so one axis is 3x longer

$X_{n\times d}$ usually isn't square!

so it doesn't have eigenvectors, can't be diagonalized, etc.



Unit Circle



Axes of ellipse are eigenvectors of A!
$\vec{v}_2$
$\vec{v}_1$
cloud of data

$X_{n \times d}$ $\qquad X^T X \qquad XX^T$

square
symmetric

"singular value decomposition"

$$X = U \Sigma V^T$$

U eigvecs of XX^T

V contains eigvecs of $X^T X$