EECS 245 Fall 2025

Math for ML

Lecture 24: Singular Value Decomposition

Read Ch. 5.3 (brand new!)

Agenda

all about the

singular value decomposition

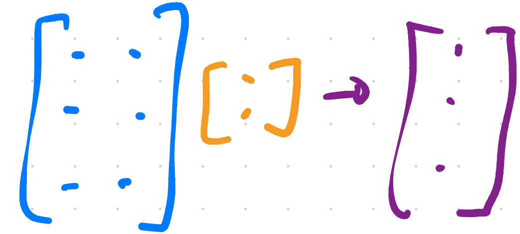
$$X = U \Sigma V^T$$

- today: mechanics of SVD
- next week: more applications of the SVD

Announcements

- HW 11 coming soon
  - short
  - due Friday 12/5
- Read MT 2 sols
- No lab tmrw, no lecture Thursday, no office hours until Monday (except right after lecture)

Eigenvalues, eigenvectors



λ $\vec{v}$

A square!

$$A\vec{v} = \lambda\vec{v}$$

$A = V\Lambda V^{-1}$ iff A diagonalizable

direction is preserved when multiplying by A

now, suppose X is an $n \times d$ matrix (not necessarily square)

$$X\underbrace{\vec{v}}_{\mathbb{R}^d} \to \sigma\vec{u}$$

$\mathbb{R}^n$

singular vectors

singular value

singular value decomposition

X is <u>any</u> $n \times d$ matrix

$$X_{n \times d} = U \Sigma V^T$$

$n \times d$ diagonal matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \end{bmatrix}$$

important: arrange U, Σ, V^T such that

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

$n \times n$ orthogonal matrix

$U^T U = U U^T = I_n$

columns of U are called "left sing. vecs"

V is a $d \times d$ orthogonal matrix

$V^T V = V V^T = I_d$

cols V are called right sing. vecs.

σ_i's sorted

$$\underbrace{\begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 5 \\ 2 & -2 & 0 \\ 5 & 5 & 10 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{2}{3} \\ \frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}}_{V^T}$$

2 non-zero singular values!

$\text{rank}(X) = 2$

rank = # of non-zero σ_i's

$$X = U\Sigma V^T$$

① $X^T X = (U\Sigma V^T)^T U\Sigma V^T$

$$= V\Sigma^T \underbrace{U^T U}_{I} \Sigma V^T$$

$$= V\underbrace{\Sigma^T\Sigma}\, V^T$$

components are σ_1^2, σ_2^2 ...

② $XX^T = U\Sigma V^T (U\Sigma V^T)^T$

$$= U\Sigma \underbrace{V^T V}_{I} \Sigma^T U^T$$

$$= U\underbrace{\Sigma\Sigma^T}\, U^T$$

same diagonal as $\Sigma^T\Sigma$!

observation: $X^T X$ and XX^T are square, and so have eigenvals/eigenvecs! also symmetric, so eigenvecs for different λ_i's are orthogonal (spectral thm) aside: if A symmetric, then

$$A = Q \Lambda Q^T$$

big takeaway: in $X = U\Sigma V^T$,

1. the col's of U are the eigenvecs of XX^T
2. the col's of V are the eigenvecs of X^TX
3. what goes in Σ? $\sqrt{\lambda_i} = \sigma_i$, where λ_i are eigvals of X^TX

$\Sigma^T \Sigma$ is $d \times d$

$\Sigma \Sigma^T$ is $n \times n$

the non-zero values in both are shared!

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{bmatrix}_{4 \times 3} \qquad \Sigma^T \Sigma = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \sigma_3^2 \end{bmatrix}_{3 \times 3}$$

$$\Sigma \Sigma^T = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \sigma_3^2 & \\ 0 & & & 0 \end{bmatrix}_{4 \times 4} \Rightarrow$$

big takeaway:

if λ_i is an eigenvalue of $X^T X$ (equivalently, $X X^T$)

then $\sigma_i = \sqrt{\lambda_i}$ is a singular value of X.

$$X = \begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 5 \\ 2 & -2 & 0 \\ 5 & 5 & 10 \end{bmatrix} \qquad X^T X = \begin{bmatrix} 42 & 33 & 75 \\ 33 & 42 & 75 \\ 75 & 75 & 150 \end{bmatrix}$$

what are eigvals, eigvecs of $X^T X$?

$\lambda_1 = 225$

$\lambda_2 = 9$

$\lambda_3 = 0$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ converted to unit vec so V orthogonal

$$\underbrace{\begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 5 \\ 2 & -2 & 0 \\ 5 & 5 & 10 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{2}{3} \\ \frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}}_{V^T}$$

$\frac{1}{15} X \vec{v}_1$ $\quad$ $\frac{1}{3} X \vec{v}_2$

recipe!

$\vec{u}_3, \vec{u}_4$ basis for $\text{nullsp}(XX^T)$

$\sqrt{\lambda_2} = \sqrt{9}$

$$X = U \Sigma V^T$$

$$X V = U \Sigma$$

$$X \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \\ | & | & | & | \end{bmatrix} \left[\begin{array}{ccc} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_3 \\ \multicolumn{3}{c}{\text{all } 0!} \end{array}\right]$$

no $\vec{u}_4$

$$\begin{bmatrix} | & | & | \\ X\vec{v}_1 & X\vec{v}_2 & X\vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \sigma_1 \vec{u}_1 & \sigma_2 \vec{u}_2 & \sigma_3 \vec{u}_3 \\ | & | & | \end{bmatrix}$$

$$\Rightarrow X \vec{v}_i = \sigma_i \vec{u}_i$$

A recipe for $\vec{u}_i$'s if you have all σ_i's, $\vec{v}_i$'s:

$$\vec{u}_i = \frac{1}{\sigma_i} \times \vec{v}_i$$

$$\vec{u}_1 = \frac{1}{15} \times \overset{\vec{v}_1}{\begin{bmatrix} 1/\sqrt{6} \\ 4/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}}$$

$$\vec{u}_2 = \frac{1}{3} \times \underset{\vec{v}_2}{\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}}$$

$$\vec{u}_3 = \frac{1}{0} \; ?$$

recipe only works for $i = 1, 2, \ldots, r$, where r = rank

$\vec{v}_i$'s basis for rowsp(X)

\# of non-zero σ_i's = rank(X)

basis for colsp(X)

$$X = \underbrace{\begin{bmatrix} | & \cdots & | & | & \cdots & | \\ \vec{u}_1 & \cdots & \vec{u}_r & \vec{u}_{r+1} & \cdots & \vec{u}_n \\ | & \cdots & | & | & \cdots & | \end{bmatrix}}_{U \quad n \times n} \underbrace{\begin{bmatrix} \sigma_1 & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & \sigma_r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\Sigma \quad n \times d} \underbrace{\begin{bmatrix} — \ \vec{v}_1^T \ — \\ \vdots \\ — \ \vec{v}_r^T \ — \\ — \ \vec{v}_{r+1}^T \ — \\ \vdots \\ — \ \vec{v}_d^T \ — \end{bmatrix}}_{V^T \quad d \times d}$$

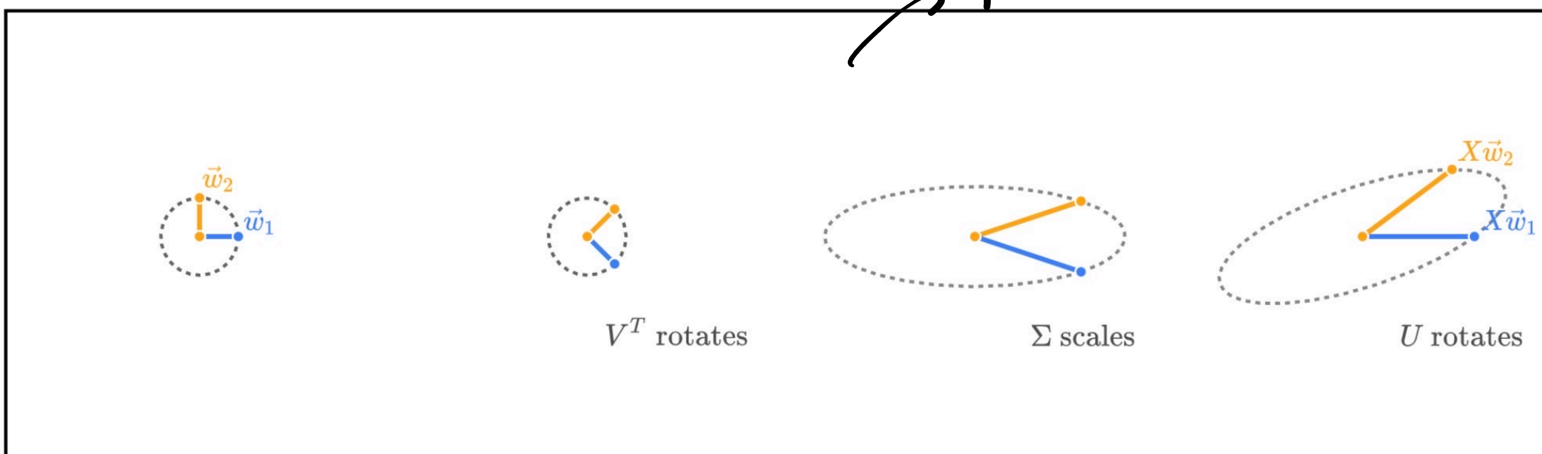
basis for nullsp(X^T)

basis for nullsp(X)

$$X = \underbrace{U}_{\text{orthogonal: rotation}}\ \underbrace{\Sigma}_{\text{diagonal: stretch}}\ \underbrace{V^T}_{\text{orthogonal: rotation}}$$

for some 2x2 X

Visualizing $X\vec{w} = U\Sigma V^T \vec{w}$ for two different vectors



$$\underbrace{\begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 5 \\ 2 & -2 & 0 \\ 5 & 5 & 10 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{2}{3} \\ \frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}}_{V^T}$$

$$= 15 \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} + 3 \begin{bmatrix} 1/3\sqrt{2} \\ \vdots \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

rank 2 = sum of 2 rank-ones!

$$\underbrace{\begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 5 \\ 2 & -2 & 0 \\ 5 & 5 & 10 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{2}{3} \\ \frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}}_{V^T}$$

The summation view of the SVD says that:

$$X = 15 \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ 0 \\ \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{2\sqrt{2}}{3} \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{5}{2} & \frac{5}{2} & 5 \\ \frac{5}{2} & \frac{5}{2} & 5 \\ 0 & 0 & 0 \\ 5 & 5 & 10 \end{bmatrix}}_{\text{rank-one matrix}} + \underbrace{\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{rank-one matrix}}$$

pretty close to X already!

Since $15 > 3$, the first outer product contributes more to X than the second one does.

see notes for low-rank approximation

full matrix:

$$X = \underbrace{\sigma_1 \vec{u}_1 \vec{v}_1^{\,T}}_{\text{each "piece" adds resolution}} + \overbrace{\sigma_2 \vec{u}_2 \vec{v}_2^{\,T}}^{\text{outer product}} + \cdots + \sigma_r \vec{u}_r \vec{v}_r^{\,T}$$

approx:

$$X_k = \sigma_1 \vec{u}_1 \vec{v}_1^{\,T} + \cdots + \sigma_k \vec{u}_k \vec{v}_k^{\,T}, \quad \text{where } k \le r$$

Rank k: 18