Lab 4: Projections, Span, and Linear Independence Solutions

EECS 245, Spring 2026 at the University of Michigan

due for completion at 11:59PM Ann Arbor Time on Monday, May 18th, 2026

Name: ____________________

uniqname: ____________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete as many of the activities as you can in 2 hours and submit a PDF of your work to Gradescope. We will provide specific instructions on how to submit programming activities (e.g. submitting the notebook or including a screenshot of some output).

Feel free to work with others in the course, but you must submit individually.

Recap: Projections, Span, and Linear Independence

- (Chapter 3.4) The **orthogonal projection** of the vector $\vec{u}$ onto the vector $\vec{v}$ is given by

$$\vec{p} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$

  Above, the scalar $k^* = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$ was chosen to minimize $\|\vec{u} - k\vec{v}\|^2$.

- The vector $\vec{p}$ is called the orthogonal projection because the resulting error vector,

$$\vec{e} = \vec{u} - \vec{p} = \vec{u} - k^*\vec{v}$$

  is orthogonal to $\vec{v}$.

$$\vec{e} \cdot \vec{v} = 0$$

- (4.1) The **span** of a set of vectors is the set of all possible linear combinations of the vectors in the set.

$$\text{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\}) = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_d\vec{v}_d \mid a_1, a_2, \ldots, a_d \in \mathbb{R}\}$$

  - The span of one vector in $\mathbb{R}^n$ is a line through the origin.
  - The span of two **non-parallel** vectors in $\mathbb{R}^n$ is a plane through the origin; this plane is called a **2-dimensional subspace** of $\mathbb{R}^n$.
  - In general, the span of d vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ of dimension 0 to d, depending on the vectors and their relationships.
  - Think of a d-dimensional subspace of $\mathbb{R}^n$ as a "slice" of $\mathbb{R}^n$ that goes through the origin, in which you can move in d directions.

Activity 1: Presidential Speeches and Cosine Similarity

Complete the tasks in the `lab04.ipynb` notebook.

There are two ways to access the supplemental Jupyter Notebook:

- **Option 1 (preferred)**: Set up a Jupyter Notebook environment locally, use `git` to clone our **course repository**, and open `labs/lab04/lab04.ipynb`. For instructions on how to do this, see the **Environment Setup** page of the course website.
- **Option 2**: Click **here** to open `lab04.ipynb` on DataHub. Before doing so, read the instructions on the **Environment Setup** page on how to use the DataHub.

Once you're done, include a screenshot of your completed Activity 1 implementation in your PDF submission of Lab 4 to Gradescope, making sure to include proof that the (local) autograder passed. Instructions on how to do this are in the lab notebook.

Activity 2: Orthogonal Projections

Let $\vec{c} = \begin{bmatrix} 1 \\ 2 \\ -4 \\ 0 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix}$. Note that $\|\vec{c}\|^2 = 21$, $\|\vec{d}\|^2 = 14$, and $\vec{c} \cdot \vec{d} = 7$.

a) Find the orthogonal projection of $\vec{c}$ onto $\vec{d}$. Call this vector $\vec{q}$.

Solution:

$$\begin{aligned}
\vec{q} &= \left(\frac{\vec{c} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\right) \vec{d} \\
&= \frac{1 \cdot 3 + 2 \cdot 2 + (-4) \cdot 0 + 0 \cdot (-1)}{3^2 + 2^2 + 0^2 + (-1)^2} \vec{d} \\
&= \frac{3 + 4}{9 + 4 + 1} \vec{d} \\
&= \frac{7}{14} \vec{d} \\
&= \frac{1}{2} \vec{d} \\
&= \begin{bmatrix} 1.5 \\ 1 \\ 0 \\ -0.5 \end{bmatrix}
\end{aligned}$$

b) Find the error vector, $\vec{r} = \vec{c} - \vec{q}$. Which vector is $\vec{r}$ orthogonal to, $\vec{c}$ or $\vec{d}$? Draw a rough picture of the relationship between $\vec{c}$, $\vec{d}$, $\vec{q}$, and $\vec{r}$.

Solution: First, let's find $\vec{r}$.

$$\begin{aligned}\vec{r} &= \vec{c} - \vec{q} \\ &= \begin{bmatrix} 1 \\ 2 \\ -4 \\ 0 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 1 \\ 0 \\ -0.5 \end{bmatrix} \\ &= \begin{bmatrix} -0.5 \\ 1 \\ -4 \\ 0.5 \end{bmatrix}\end{aligned}$$

$\vec{r}$ is orthogonal to $\vec{d}$, not $\vec{c}$, as confirmed by the dot products. The key idea we introduced in **Chapter 3.4** is that the error vector is orthogonal to the vector we projected onto. Here, $\vec{r}$ is the error vector and $\vec{d}$ is the vector we projected onto.

$$\begin{aligned}\vec{r} \cdot \vec{c} &= (-0.5) \cdot 1 + 1 \cdot 2 + (-4) \cdot (-4) + 0.5 \cdot 0 \\ &= -0.5 + 2 + 16 \\ &= 17.5\end{aligned}$$

$$\begin{aligned}\vec{r} \cdot \vec{d} &= (-0.5) \cdot 3 + 1 \cdot 2 + (-4) \cdot 0 + 0.5 \cdot (-1) \\ &= -1.5 + 2 - 0.5 \\ &= 0\end{aligned}$$

Activity 3: Orthogonal Decomposition

a) Let $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Write $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, and verify that your answer is correct. Note that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are pairwise orthogonal.

Solution:
We are given

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \quad \vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We're looking for scalars a, b, c such that $\vec{u} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$.

Solution 1: Solving a system of equations

$$a \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + b \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + c \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Is equivalent to the system of equations:

$$-a + 2b + 2c = 1 \tag{1}$$
$$2a + 2b - c = 1 \tag{2}$$
$$2a - b + 2c = 1 \tag{3}$$

Using elimination:

$$\begin{aligned} \text{(Eq. 2)} - \text{(Eq. 3)}: \quad & (2a - 2a) + (2b - (-b)) + (-c - 2c) = 0 \\ & \Rightarrow \quad 3b - 3c = 0 \quad \Rightarrow \quad b = c \\ \text{(Eq. 2)} - \text{(Eq. 1)}: \quad & (2a - (-a)) + (2b - 2b) + (-c - 2c) = 0 \\ & \Rightarrow \quad 3a - 3c = 0 \quad \Rightarrow \quad a = c \end{aligned}$$

This tells us that $a = b = c$. Plugging this back into Eq. 2 gives us:

$$-a + 2a + 2a = 1 \to 3a = 1 \to a = \frac{1}{3}$$

So, $a = b = c = \frac{1}{3}$, and:

$$\vec{u} = \tfrac{1}{3}\,\vec{v}_1 + \tfrac{1}{3}\,\vec{v}_2 + \tfrac{1}{3}\,\vec{v}_3$$

We can verify that we did this correctly by computing the right-hand side above:

$$\frac{1}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_2 + \frac{1}{3}\vec{v}_3 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/3 + 2/3 + 2/3 \\ 2/3 + 2/3 - 1/3 \\ 2/3 - 1/3 + 2/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{u}$$

Solution 2: Using the fact that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthogonal
As is alluded to in part **b)**, we can use the fact that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthogonal to find coefficients a, b, and c by projecting $\vec{u}$ onto each of the $\vec{v}_i$s. This is similar to what was done in the **Orthogonal Decomposition** section of Chapter 3.4.

Let $\vec{p}_i$ be the projection of $\vec{u}$ onto $\vec{v}_i$, for $i = 1, 2, 3$. Then, we have:

$$\vec{p}_1 = \frac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \frac{1 \cdot (-1) + 1 \cdot 2 + 1 \cdot 2}{(-1)^2 + 2^2 + 2^2}\vec{v}_1 = \frac{3}{9}\vec{v}_1 = \frac{1}{3}\vec{v}_1$$

$$\vec{p}_2 = \frac{\vec{u} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = \frac{1 \cdot 2 + 1 \cdot 2 + 1 \cdot (-1)}{2^2 + 2^2 + (-1)^2}\vec{v}_2 = \frac{3}{9}\vec{v}_2 = \frac{1}{3}\vec{v}_2$$

$$\vec{p}_3 = \frac{\vec{u} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3}\vec{v}_3 = \frac{1 \cdot 2 + 1 \cdot (-1) + 1 \cdot 2}{2^2 + (-1)^2 + 2^2}\vec{v}_3 = \frac{3}{9}\vec{v}_3 = \frac{1}{3}\vec{v}_3$$

Adding $\vec{p}_1, \vec{p}_2, \vec{p}_3$ gives us:

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = \frac{1}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_2 + \frac{1}{3}\vec{v}_3 = \vec{u}$$

b) In general, suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are **orthogonal** vectors in $\mathbb{R}^n$, meaning that $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$. **Given that** it is possible to write $\vec{u}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$, show that the coefficients of the linear combination

$$\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_d\vec{v}_d$$

are given by

$$a_i = \frac{\vec{u} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

Hint: Start by taking the dot product of both sides of the linear combination equation with $\vec{v}_1$. What do you notice?

Solution:
We're told to assume that any pair of vectors among $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are orthogonal, and that $\vec{u}$ can be written as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$.

$$\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_d\vec{v}_d$$

As the hint suggests, let's take the dot product of both sides with $\vec{v}_i$, where i is some value in $\{1, 2, \ldots, d\}$.

$$\vec{u} \cdot \vec{v}_i = \big(a_1\vec{v}_1 + \cdots + a_d\vec{v}_d\big) \cdot \vec{v}_i$$

Since $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$, only the $i = j$ term survives:

$$\begin{aligned}
\vec{u} \cdot \vec{v}_i &= \big(a_1\vec{v}_1 + \cdots + a_d\vec{v}_d\big) \cdot \vec{v}_i \\
&= a_1(\vec{v}_1 \cdot \vec{v}_i) + \cdots + a_{i-1}(\vec{v}_{i-1} \cdot \vec{v}_i) + a_i(\vec{v}_i \cdot \vec{v}_i) + a_{i+1}(\vec{v}_{i+1} \cdot \vec{v}_i) + \cdots + a_d(\vec{v}_d \cdot \vec{v}_i) \\
&= a_1(0) + \cdots + a_{i-1}(0) + a_i(\vec{v}_i \cdot \vec{v}_i) + a_{i+1}(0) + \cdots + a_d(0) \\
&= a_i(\vec{v}_i \cdot \vec{v}_i)
\end{aligned}$$

Solving for a_i above gives us

$$\vec{u} \cdot \vec{v}_i = a_i(\vec{v}_i \cdot \vec{v}_i) \implies a_i = \frac{\vec{u} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

Since i was arbitrary, the same calculation holds for any value of i in $\{1, 2, \ldots, d\}$.
What we proved here in part **b)** is that when writing a vector $\vec{u}$ as a linear combination of orthogonal vectors, the coefficients of the linear combination can be found by projecting the vector $\vec{u}$ onto each of the orthogonal vectors and adding the results, rather than solving a system of equations.

Activity 4: Planes and the Cross Product

An important idea from **Chapter 4.1** is that two non-parallel vectors in $\mathbb{R}^n$ (where $n \geq 2$) span a plane in n-dimensional space. Here, we'll show you how to find the equation of such a plane, given two vectors in $\mathbb{R}^3$. This is also touched on in **Chapter 4.4**.

a) Given two vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$, show that the vector $\vec{q}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

$$\vec{q} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

The vector $\vec{q}$ is called the **cross product** of $\vec{a}$ and $\vec{b}$. The cross product is only defined for two vectors in $\mathbb{R}^3$ specifically, and the product is another vector in $\mathbb{R}^3$. (This differentiates it from the dot product, which is defined for two vectors in any $\mathbb{R}^n$, and whose output is a scalar.)

Solution:

$$\begin{aligned} \vec{a} \cdot \vec{q} &= a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) \\ &= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_1a_2b_3 + a_1a_3b_2 - a_2a_3b_1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{b} \cdot \vec{q} &= b_1(a_2b_3 - a_3b_2) + b_2(a_3b_1 - a_1b_3) + b_3(a_1b_2 - a_2b_1) \\ &= a_2b_1b_3 - a_3b_1b_2 + a_3b_1b_2 - a_1b_2b_3 + a_1b_2b_3 - a_2b_1b_3 \\ &= 0 \end{aligned}$$

b) Find the cross product of $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

Solution:
Let's call the cross product of $\vec{v}_1$ and $\vec{v}_2$ $\vec{q}$.

$$\begin{aligned} \vec{q} &= \begin{bmatrix} (-1)\cdot(-1) - 3\cdot 2 \\ 3\cdot 1 - 2\cdot(-1) \\ 2\cdot 2 - (-1)\cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-6 \\ 3+2 \\ 4+1 \end{bmatrix} \\ &= \begin{bmatrix} -5 \\ 5 \\ 5 \end{bmatrix} \end{aligned}$$

c) Let $\vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$ be your answer to part **b)**.

Verify that the points $(0, 0, 0)$, $(2, -1, 3)$ and $(1, 2, -1)$ satisfy the equation

$$q_1 x + q_2 y + q_3 z = 0$$

(Those points are the endpoints of the vectors $\vec{v}_1$ and $\vec{v}_2$, along with the origin.)

Solution:
The equation of the plane in question is

$$-5x + 5y + 5z = 0$$

For $(0, 0, 0)$,

$$-5(0) + 5(0) + 5(0) = 0$$

For $(2, -1, 3)$,

$$\begin{aligned} &-5(2) + 5(-1) + 5(3) \\ &= -10 - 5 + 15 \\ &= 0 \end{aligned}$$

For $(1, 2, -1)$,

$$\begin{aligned} &-5(1) + 5(2) + 5(-1) \\ &= -5 + 10 - 5 \\ &= 0 \end{aligned}$$

Note that a simpler way of writing the equation of the plane is $x - y - z = 0$, which comes from dividing both sides of the original equation by -5. Another equivalent form is $z = x - y$.

d) Above, we wrote the equation of the plane spanned by $\vec{v}_1$ and $\vec{v}_2$ in the "standard form" for planes in $\mathbb{R}^3$, $ax + by + cz + d = 0$ (where $d = 0$). Now, write the equation of the plane spanned by $\vec{v}_1$ and $\vec{v}_2$ in **parametric** form. The parametric form of a plane is given by

$$P = \vec{p}_0 + s\vec{u} + t\vec{v}, \quad s, t \in \mathbb{R}$$

This won't require much work; it's more that we want you to understand that there are two ways of expressing planes in $\mathbb{R}^3$. In higher dimensions, all planes (also called 2-dimensional subspaces) must be expressed in parametric form. Read **Chapter 4.4**.

Solution: $P = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad s_1, s_2 \in \mathbb{R}$

The plane spanned by $\vec{v_1}$ and $\vec{v_2}$ is the set of all of their possible linear combinations, shown through the scalars s_1 and s_2. The $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ at the start isn't necessary to write, since if we set $s_1 = s_2 = 0$ we get the point $(0, 0, 0)$ anyways, but it's good to emphasize that the span of $\vec{v_1}$ and $\vec{v_2}$ includes the origin. **The span of any collection of vectors always includes the origin.**

Activity 5: Finding a Linearly Independent Subset

Recall from Chapter 4.2 that a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ is **linearly independent** if either of the following equivalent conditions hold:

- None of the vectors can be written as a linear combination of the others.
- The only way to create the zero vector as a linear combination of the vectors is if all the coefficients are zero. In other words, the only solution to

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d = \vec{0}$$

  is $a_1 = a_2 = \ldots = a_d = 0$.

Chapter 4.2 introduces an algorithm for finding a linearly independent subset of a given set of vectors with the same span as the original set:

```
given v_1, v_2, ..., v_d
initialize linearly independent set S = {v_1}
for i = 2 to d:
    if v_i is not a linear combination of S:
        add v_i to S
```

In each of the parts below, find **a linearly independent** set of vectors that spans the same span as the given set of vectors. There are multiple possible answers for each part, but all of them have the same number of vectors.

a)

$$\vec{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix}2\\3\\4\end{bmatrix}$$

Solution: $i = 2, S = \{\vec{v_1}\} : \vec{v_2} \notin \text{span}(S)$. There's no way to make $\vec{v_2}$ as a linear combination of v_1, because $\vec{v_1}$ has a 0 in the second component, while $\vec{v_2}$ has a 1. So, we add $\vec{v_2}$ to the set.

$i = 3, S = \{\vec{v_1}, \vec{v_2}\} : \vec{v_3} \notin \text{span}(S)$. Similar idea here, but with the third component instead of the second. Add $\vec{v_3}$ to the set.

$i = 4, S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\} : \vec{v_4} \in \text{span}(S)$. We can make $\vec{v_4}$ using $-v_1 + -v_2 + 4\vec{v_3}$.

Therefore, $\boxed{S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}}$. This is a set of 3 vectors. There are other possible answers too, like $S = \{\vec{v_1}, \vec{v_2}, \vec{v_4}\}$; all possible answers have 3 vectors. In fact, since $\text{span}(\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}) = \mathbb{R}^3$, any three vectors in $\mathbb{R}^3$ that don't all lie on the same plane will form a linearly independent set with the same span as $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

b)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Hint: Use the 0's in the vectors strategically, plus use the fact that you can't have more than 4 linearly independent vectors in $\mathbb{R}^4$.

Solution:

$$\begin{aligned}
\text{Start with } S &= \{\vec{v_1}\} \\
i = 2 &: \vec{v_2} \notin \text{span}(S),\ \ S = \{\vec{v_1}, \vec{v_2}\} \\
i = 3 &: \vec{v_3} \notin \text{span}(S),\ \ S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\} \\
i = 4 &: \vec{v_4} \in \text{span}(S),\ \ -\vec{v_1} + \vec{v_2} = \vec{v_4} \\
i = 5 &: \vec{v_5} \in \text{span}(S),\ \ -\vec{v_1} + \vec{v_3} = \vec{v_5} \\
i = 6 &: \vec{v_6} \in \text{span}(S),\ \ -\vec{v_2} + \vec{v_3} = \vec{v_6}
\end{aligned}$$

$$\text{Therefore: } \boxed{S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}}.$$

This is a set of 3 vectors.