Lab 6: Rank, Column Space, Null Space, and Inverses

EECS 245, Spring 2026 at the University of Michigan

due for completion at 11:59PM Ann Arbor Time on Wednesday, May 27th, 2026

Name: ____________________

uniqname: ____________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete as many of the activities as you can in 2 hours and submit a PDF of your work to Gradescope. We will provide specific instructions on how to submit programming activities (e.g. submitting the notebook or including a screenshot of some output).

Feel free to work with others in the course, but you must submit individually.

Note: You may find it helpful to work on the first few problems of Homework 5 before starting this lab.

Recap: Rank, Column Space, and Null Space (Chapter 5.3 and 5.4)

Suppose A is an $n \times d$ matrix. Then, rank(A) is the number of linearly independent columns of A.

	Notation	Description	Dimension	Subspace of
Column space	colsp(A)	Span of the columns of A	rank(A)	$\mathbb{R}^n$
Row space	colsp(A^T)	Span of the rows of A	rank(A)	$\mathbb{R}^d$
Null space	nullsp(A)	Set of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$	$d -$ rank(A)	$\mathbb{R}^d$

Additionally, note that you can write the dot product of two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ as $\vec{u}^T\vec{v}$:

$$\vec{u}^T = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\vec{u}^T\vec{v} = u_1v_1 + \cdots + u_nv_n = \sum_{i=1}^{n}(u_iv_i) = \vec{u} \cdot \vec{v} \qquad \textbf{(not } \vec{u}^T \cdot \vec{v}\textbf{)}$$

Activity 1: Null Space of a Matrix with Linearly Independent Columns

Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \\ 1 & 0 \end{bmatrix}$. What is $\text{nullsp}(A)$?

Activity 2: Fundamentals

Let $X = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 & 4 \\ 2 & 5 & -2 & 7 & 11 & 10 \\ 4 & 8 & -4 & 12 & 16 & 16 \end{bmatrix}$.

a) Find a basis for $\text{colsp}(X)$. What is $\text{rank}(X)$? Why? *Hint: Column 5 is a linear combination of columns 1 and 2. With this fact, you should be able to answer this relatively quickly.*

b) Fill in the blanks: $\text{colsp}(X^T)$ is a ____-dimensional subspace of ____.

c) Fill in the blanks: $\text{nullsp}(X)$ is a ____-dimensional subspace of ____.

d) Find a basis for $\text{nullsp}(X)$. *Hint: You should be able to answer this without solving equations.*

Suppose A is an $n \times d$ matrix with rank r. A CR decomposition of A is a product of two matrices C and R, where $A = CR$ and:

- C is an $n \times r$ matrix and R is a $r \times d$ matrix
- C contains the linearly independent columns of A, selected left-to-right
- R tells how to "mix" the columns of C (which are linearly independent) to reconstruct the columns of A

Let's keep working with $X = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 & 4 \\ 2 & 5 & -2 & 7 & 11 & 10 \\ 4 & 8 & -4 & 12 & 16 & 16 \end{bmatrix}$.

e) Find a CR decomposition of X. This shouldn't take very much work; review your work from part **a)** in finding a basis for $\text{colsp}(X)$.

The key idea being assessed here is that in $A = CR$, the columns of C are linearly independent and a basis for $\text{colsp}(A)$, while the rows of R are linearly independent and a basis for $\text{colsp}(A^T)$!

Activity 3: Outer Products

Suppose $A = \vec{u}\vec{v}^T + \vec{w}\vec{z}^T$, where $\vec{u}, \vec{v}, \vec{w}, \vec{z} \in \mathbb{R}^n$ are non-zero vectors.

a) What is $\text{rank}(\vec{u}\vec{v}^T)$?

b) Under what conditions is $\text{rank}(A) = 2$? What about $\text{rank}(A) < 2$? *Hint: First, think about what happens when multiplying A by a vector $\vec{x} \in \mathbb{R}^n$. Can you write this as a linear combination of some other vectors? The case for $\text{rank}(A) = 2$ is more complicated than "$\vec{u}$ and $\vec{w}$ are linearly independent."*

Activity 4: The Systems of Equations View

Let A be an $n \times d$ matrix of rank r, and suppose there exists vectors $\vec{b} \in \mathbb{R}^n$ such that

$$A\vec{x} = \vec{b}$$

does not have a solution (meaning no $\vec{x}$ makes $A\vec{x} = \vec{b}$).

a) What are all inequalities ($<$ or $\leq$) that must be true between n, d, and r?

b) How do you know that $A^T\vec{y} = \vec{0}$ has solutions other than $\vec{y} = \vec{0}$?

Activity 5: Symbolic Inverses

Given that A is an invertible $n \times n$ matrix that satisfies $A^4 - 3A^2 + 2A - 4I = 0$, find an expression for A^{-1} in terms of A.

Activity 6: Basics of Invertibility

Suppose A is an $n \times n$ matrix. State as many of the equivalent conditions for invertibility as you can.

Activity 7: Programming

Complete the tasks in the `lab06.ipynb` notebook. Watch **this video** first with tips on using `numpy` for linear algebra.

There are two ways to access the supplemental Jupyter Notebook:

- **Option 1 (preferred)**: Set up a Jupyter Notebook environment locally, use `git` to clone our **course repository**, and open `labs/lab06/lab06.ipynb`. For instructions on how to do this, see the **Environment Setup** page of the course website.
- **Option 2**: Click **here** to open `lab06.ipynb` on DataHub. Before doing so, read the instructions on the **Environment Setup** page on how to use the DataHub.

Once you're done, include a screenshot of your completed Activity 7 implementation in your PDF submission of Lab 6 to Gradescope, making sure to include proof that the (local) autograder passed. Instructions on how to do this are in the lab notebook.